TMA4195 - MATHEMATICAL MODELING (FALL 2017).

MODELING OF HEAT PIPES

Heat pipes are autonomous devices developed to transport heat very efficiently and without requiring external work. The first targeted applications were spacecraft equipment but nowadays they are used in almost every computer to transfer the heat produced by the processors to the cooling system, see Figure 1. As many

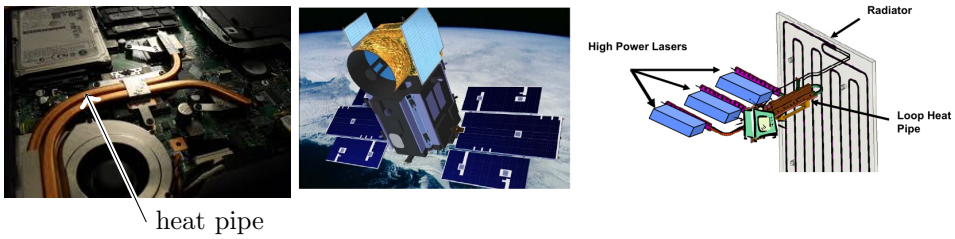

FIGURE 1. Heat pipes are essential components of space equipment and computers.

other cooling devices, heat pipes make use of the latent heat, which is the heat that is absorbed or released by a fluid when it goes through a phase transition. The mechanism is illustrated in Figure 2. A heat pipe can be represented as a closed tube which contains a fluid and where the wall are covered with a porous material called the *wick*, see Figure 3. On one end of the tube, the heat pipe absorbs external heat. The liquid in the wick is warmed up, turned into gas and leaves the wick towards the empty space at the center of the pipe. The gas flows through the pipe and, at the other end, condensates in contact with a heat sink. The fluid then goes back into the wick structure in a liquid form. The system is closed and there is no external mechanism, such as a pump, that drives the liquid back the heated end. The liquid is moved by the capillary forces alone, which are created in the porous structure of the wick. According to wikipedia [1], heat pipes are 500 times much more efficient as copper to transfer heat.

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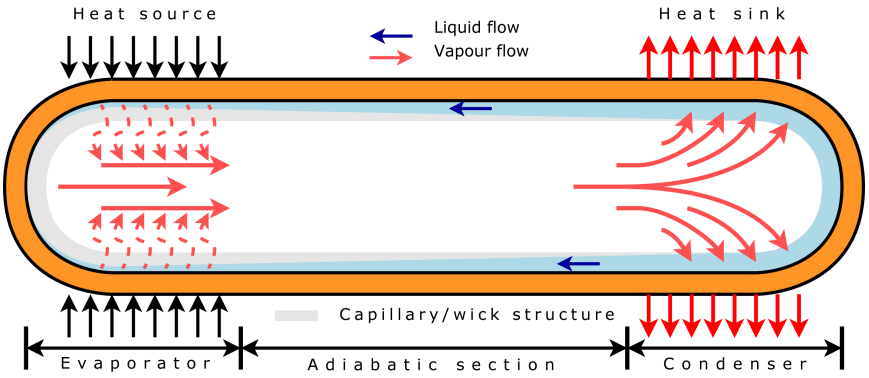

FIGURE 2. Illustration of the heat pipe mechanism

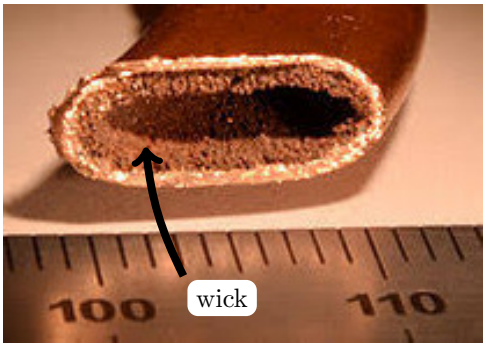

FIGURE 3. cross-section of a heat pipe showing the wick structure.

General governing equations

Let us derive the governing equations in the gas region. We denote by ρ the density and by $u \in \mathbb{R}^3$ the velocity.

Question: ***By using a control volume and letting it tend to a point, show that the mass conservation equation is given by***

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0. \tag{1}$$

Let us neglect gravity and the viscous force for the moment. Then, the pressure forces are the unique forces in the system.

Question: ***In the same way as for the mass conservation equation, derive the momentum conservation equation,***

$$(\rho u)_t + \nabla \cdot (\rho u \otimes u) = -\nabla p, \tag{2}$$

that is

$$(\rho u_1)_t + \nabla \cdot (\rho u_1 u) = -\frac{\partial p}{\partial x},$$
$$(\rho u_2)_t + \nabla \cdot (\rho u_2 u) = -\frac{\partial p}{\partial y},$$
$$(\rho u_3)_t + \nabla \cdot (\rho u_3 u) = -\frac{\partial p}{\partial z}.$$

It remains to set up the equation for the energy. We know from thermodynamics that any change in the internal energy of a system denoted δU is equal to the sum of the work exerted on the system δW and the heat absorbed by the system δQ,

$$\delta U = \delta W + \delta Q.$$

In our case, for a given control volume, work will be exerted on the surface of the control volume by the pressure forces of the surrounding fluid. Heat will be transferred from the surrounding fluid to the control volume by a heat flux on the surface. We will use Fourier's law, which states that the heat flux q (in $\mathrm{W m^{-2}}$) is proportional to the temperature gradient,

$$q = -\kappa \nabla T.$$

Above, T denotes the temperature (in K) and κ denotes the thermal conductivity (in $\mathrm{W\,m^{-1}\,K^{-1}}$). In addition, the system has kinetic energy. We denote by e the specific internal energy density (in $\mathrm{W\,kg^{-1}}$) which is internal energy per unit of mass. The specific enthalpy h is defined as

$$h = e + \frac{p}{\rho}$$

Question: ***Using the same method as before, show that the conservation of energy equation is given by***

$$(\rho(e + \frac{1}{2}u^2))_t + \nabla \cdot (\rho u(\frac{1}{2}u^2 + h) - \kappa \nabla T) = 0, \tag{3}$$

Heat transfer using latent heat

We set up a model to show how heat can be transported very efficiently, with very low temperature gradient, using latent heat. In this case, we will in fact assume that the temperature is constant, say $T = 100\,^{\circ}\mathrm{C}$. We consider a one-dimensional

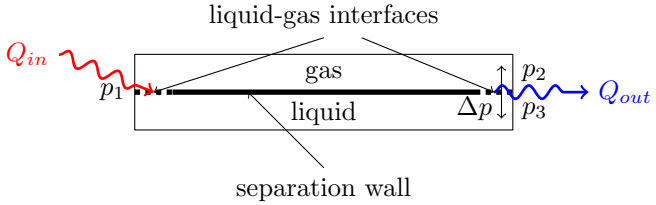

FIGURE 4. One dimensional model

geometry, see Figure 4. In the upper channel, we have the gas flowing from left to right and in the lower channel the liquid flowing in the opposite direction. To simplify the computations, we assume the the fluids are incompressible and we replace

the momentum equation by a Darcy approximation, where the flux is proportional to the pressure gradient,

$$u = -k\nabla p,$$

for some constant k. On the left-hand side, we have that the vapor and liquid are at equilibrium, that is

$$p = p_{\text{sat}}(T),$$

see Figure 5. On the right-hand side, we assume that we have a condensing device, which we do not detail and which guarantees a given pressure jump so that a cycling flow can be obtained and that the gas is transformed to liquid and moved back in the liquid channel. To obtain the enthalpy $h(p, T)$ of water, we can use the database provided by the National Institute of Standards and Technology ([2]) . We consider the steady state solution.

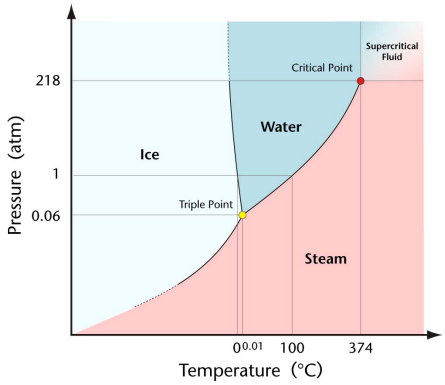

FIGURE 5. Phase diagram for water

Question: ***Under these assumptions, compute the efficiency of the heat pipe by computing the ratio between the amount of heat that is loss along the wall with the amount of heat that is transported.***

Gas motion generated by heat

The goal of this simplified model is to produce some mathematical evidence of the well-accepted fact that, when we warm up a gas, the gas expands, the pressure increases and it starts moving. This process occurs when the heat pipe starts working and before the steady state regime is reached.

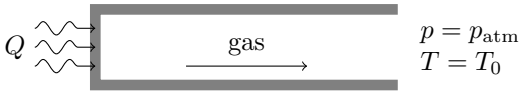

FIGURE 6. Gas flow in a heated channel

We consider a one-dimensional model. We assume that we have an ideal gas. For such gas, the pressure, density and temperature satisfy the relation

$$p = \rho RT$$

and the specific energy depends only on the temperature and is given by

$$e = c_v T,$$

where the constant c_v is the heat capacity. Initially, we have $T = T_0$, $p = p_{atm}$ and $u = 0$ in the whole tube. We assume that the problem is one-dimensional.

First approach:

Question: ***Derive the linearized equation for the temperature and try to solve it.***

Question: ***Linearize the equations for mass and momentum conservation and, by using the approximation of the temperature computed previously, try to solve them.***

Second approach: Rescale the equation in time and space, using the scaling $\tilde{t} = \varepsilon t$ and $x = \sqrt{\varepsilon} x$. This scaling is sometimes called *parabolic scaling* (Can you explain why=). See exercise below on how to proceed with scaling of variables. Then consider expansions of the unknown functions (expressed in the rescaled variables) of the form

$$\rho = \rho_0 + \varepsilon^\alpha \rho_1, \qquad u = u_0 + \varepsilon^\beta u_1, \qquad T = T_0 + \varepsilon^\delta T_1.$$

By introducing these expansions, terms are going to decouple and become easier to compute. Still, we have to keep enough interactions between terms to get interesting solutions.

Question: ***Find out the values of α, β and δ that brings interesting solutions and compute the terms in the expansions.***

Heat transfer in a steady gas flow

In this simplified model, we investigate how much heat can be remove by flowing an ideal gas. We consider the simplified one dimensional configuration presented in Figure 7. The boundary data that are given are p_l, T_l, p_r, T_r. We want to compute

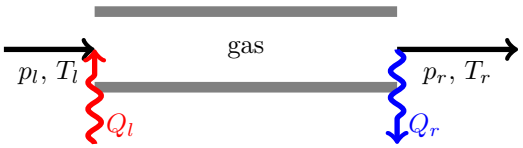

FIGURE 7. Steady gas flow in a channel

Q_l. We consider the stationary state and assume that the wall are adiabatic (no heat exchange occurs along them).

Question: ***Set up the stationary state equations. Compute the value of Q_l as a function of the boundary data.***

Interface mass exchange

In this model, we incorporate the mass and heat exchanges between the vapor and the liquid. We will not model the transport in the liquid. We assume simply that when liquid is inserted at some place it can go out anywhere else without requiring any extra work.

The liquid to vapor flux is in the vertical direction. To impose such restriction, we have to include the viscous effects. On a control volume, the viscous forces exerted

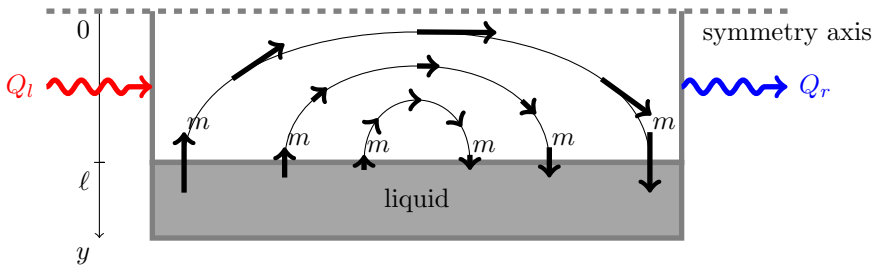

FIGURE 8. gas flow with interface mass exchange

by the surrounding fluid on the surface are given by

$$f_{\text{diss}} = \lambda \operatorname{tr}(\varepsilon)\mathbf{n} + 2\mu\varepsilon\mathbf{n} \tag{4}$$

where $\varepsilon = \frac{1}{2}(\nabla(u)+\nabla(u)^t)$ and $\mathbf{n}$ denotes the normal to the surface. It is convenient to rewrite f_{diss} as

$$f_{\text{diss}} = \sigma(\varepsilon)\mathbf{n},$$

where σ is a matrix whose definition follows from (4). The conservation of mass equation is unchanged.

Question: ***Show, in the same way as for*** (2)***, that the conservation of momentum equation is given by***

$$(\rho u)_t + \nabla \cdot (\rho u \otimes u) = -\nabla p + \lambda \nabla \operatorname{tr}(\varepsilon) + 2\mu \nabla \cdot \varepsilon. \tag{5}$$

Question: ***Show that the conservation of energy equation is given by***

$$(\rho(e + \frac{1}{2}u^2))_t + \nabla \cdot \left(\rho u(h + \frac{1}{2}u^2) + \sigma(\varepsilon)u + q\right) = 0, \tag{6}$$

Question: ***Derive the boundary conditions and the liquid-vapor interface conditions.***

One-dimensional model with mass exchange

The system of equations (1), (5) and (6) with mass exchange conditions at the interface cannot be solved analytically. We simplify it by adopting the one-dimensional configuration depicted in Figure 9. We consider the steady state regime. On the left, a heat source is applied to the liquid, at the interface, and transforms the liquid into gas, which starts flowing to the right. The quantity m in $\mathrm{kg\,s^{-1}}$ denotes the mass flow rate of liquid at the interface. At the other end, the gas condensates and is absorbed into the liquid. We assume that the interfaces are immobile. It implies that liquid is steady filled in on the left and removed on the right, but we do not model this part.

We simplify further the equations by replacing the momentum equation with a Darcy approximation. The Darcy approximation can be seen in this context as an approximation of the flow using the Poiseuille equation, see [3]. We assume that the phases are at equilibrium on both ends, meaning that the pressure at those points satisfy

$$p = p_{\text{sat}}(T),$$

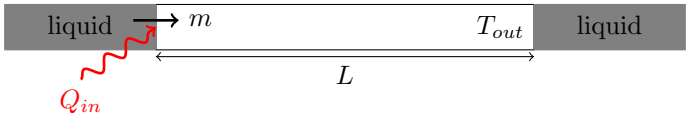

FIGURE 9. One-dimensional model with mass exchange at the interfaces

see the phase diagram in Figure 5. We assume that enthalpy jump from liquid to gas phase, denoted h_{lv}, is a constant which does not depend on temperature or pressure. In order to be able to compute the solutions analytically, let us first neglect the contribution of the viscous forces and kinetic energy in the energy equation, which then reduces to

$$(\rho e)_t + \nabla \cdot (\rho u h - \kappa \nabla T) = 0.$$

Question: ***Given Q_{in} and T_{out}, try to compute the interface mass exchange rate m and the temperature T_{in} at the inlet on the liquid-vapor interface.***

Two-dimensional model with mass exchange

Let us now go back to the two dimensional model depicted in Figure 8. However, we assume that the ratio between the width and length, denoted ε, is very small. We look at the stationary case. We assume again that viscous forces dominate and we neglect the term $\rho u \otimes u$ which comes from the inertial forces. We take $\lambda = 0$. We change the notation and denote by u and v the horizontal and vertical components of the velocity.

Question: ***Show that the mass and momentum equations simplify to***

$$\mu(2u_{xx} + u_{yy} + v_{xy}) - p_x = 0 \tag{7a}$$

$$\mu(2v_{yy} + v_{xx} + u_{xy}) - p_y = 0 \tag{7b}$$

$$(\rho u)_x + (\rho v)_y = 0 \tag{7c}$$

Question: ***Show that, if we assume that the fluid incompressible, we obtain the Stokes' equations,***

$$\begin{aligned} \mu(u_{xx} + u_{yy}) - p_x &= 0 \\ \mu(v_{yy} + v_{xx}) - p_y &= 0 \\ u_x + v_y &= 0 \end{aligned}$$

We change the spatial variable to

$$\tilde{x} = x, \quad \tilde{y} = \frac{y}{\varepsilon},$$

and introduce $\tilde{u}$ defined as

$$\tilde{u}(\tilde{x}, \tilde{y}) = u(x, y).$$

Similarly we define $\tilde{v}$ and $\tilde{p}$. We simplify the system (7) by using the incompressibility condition in the two first equations and consider the system of equations

$$\mu(u_{xx} + u_{yy}) - p_x = 0, \tag{8a}$$

$$\mu(v_{yy} + v_{xx}) - p_y = 0, \tag{8b}$$

$$(\rho u)_x + (\rho v)_y = 0. \tag{8c}$$

Note that this simplification is rather artificial. It would be interesting to solve this exercise using the original system (7).

Question: ***Rewrite the system of equations*** (8) ***using the rescaled variables*** $\tilde{u}$, $\tilde{p}$ ***and*** $\tilde{\rho}$.

In the energy equation, we remove the heat generated by the viscous forces

Question: ***Set up the energy conservation equation in this case and derive also its rescaled form.***

Only by looking at Figure 8, we can guess that the velocity field is going to be highly non-linear. To capture such solution, high order polynomial approximations would be required. We simplify the problem by assuming that there is an underlying flow from left to right, as depicted in Figure 10. It means that on the lateral walls we allow for a non-zero velocity.

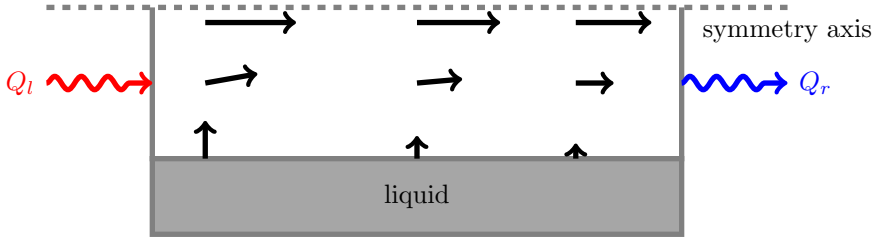

FIGURE 10. gas flow with interface mass exchange allowing horizontal flow

We remove the tilde on the variables and use expansions of the form,

$$u(x, y) = u_0(x, y) + \varepsilon u_1(x, y) + \varepsilon^2 u_2(x, y) + \dots$$

with similar expressions for the variables v, p and ρ.

Question: ***Compute the first terms of these approximations.***

Numerical simulation of a simplified 2D model

Let us set up numerical simulations of the model given in Figure 10. For this part, you need to have an implementation a Poisson solver on a rectangular geometry able to handle the classical type of boundary conditions, that is, Dirichlet or Neumann conditions. By poisson solver, we mean a numerical method to solve the Laplace equation $\Delta p = 0$. The code would also require an adjustment to handle the convection term in the energy equation, see below.

We simplify further the problem by replacing the momentum equation with a Darcy approximation, that is,

$$u = -k\nabla p$$

for some constant k. Moreover, we assume that the gas is incompressible so that we end up with the Laplace equation

$$\Delta p = 0. \tag{9}$$

The energy equation is given

$$\rho c_v u \cdot \nabla T - \kappa \Delta T = 0. \tag{10}$$

The interface conditions at $y = \ell$ are given by

$$\rho k \frac{\partial p}{\partial y} h_{lv} + \kappa \frac{\partial T}{\partial y} = 0 \tag{11}$$

Question: ***Explain equation*** (10) ***and the condition*** (11)

The system of equations is nonlinear. It can be solved by using a sequential approach. Given the pressure, we compute the temperature by solving (10). Then, we use this temperature to update the pressure by solving (9). We proceed until convergence.

Question: ***Use a sequential approach to compute a numerical approximation of the solution.***

Computation of the shape of a meniscus

The vapor and liquid phases are immiscible and they will naturally tend to minimize their interface area. In a tube, this effect results in the formation of a convex interface, as depicted in Figure 11. In this part, we want to compute the shape of the meniscus. There are three types of forces

- The gravity force,
- The surface tension between air and water, which acts such that the water molecules want to minimize the interface area,
- The attraction forces that result from the affinity between water and the container.

We consider a two dimensional configuration and denote by $f(x)$ the height of water above the horizontal plane, see Figure 11.

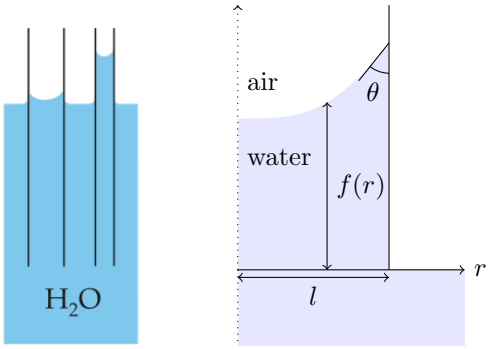

FIGURE 11. Plot of a meniscus

For each of the forces listed above, there corresponds an energy given by

$$E_g = \int_0^l \int_0^{f(x)} \rho g z \, dz dx,$$
$$E_s = \alpha \int_0^l (1 + f'(x)^2)^{\frac{1}{2}} \, dx,$$
$$E_c = \beta f(l),$$

where α and β are constant coefficients measuring the strength of the gas-liquid and liquid-solid interaction forces.

Question: ***Explain how these expressions have been derived.***

The total energy is given by

$$E = E_g + E_s - E_c$$

and the shape of the meniscus is obtain by finding the function f which minimizes E.

Question: ***Use calculus of variation techniques to derive the differential equations that the minimizing function f satisfies.***

Derivation of Young-Laplace's equation

At a gas-liquid interface, the surface tension induces a pressure jump between the two phases. The value of the pressure jump is given by the Young-Laplace equation,

$$(p_v - p_l) = \sigma \left(\frac{1}{R_1} + \frac{1}{R_2} \right). \tag{12}$$

Above, R_1 and R_2 denotes the curvature radii, p_v and p_l denote the vapor and liquid pressure, on their respective sides of the interface. The surface tension coefficient is given by the constant σ. For any point O, up to a rotation and a translation, the

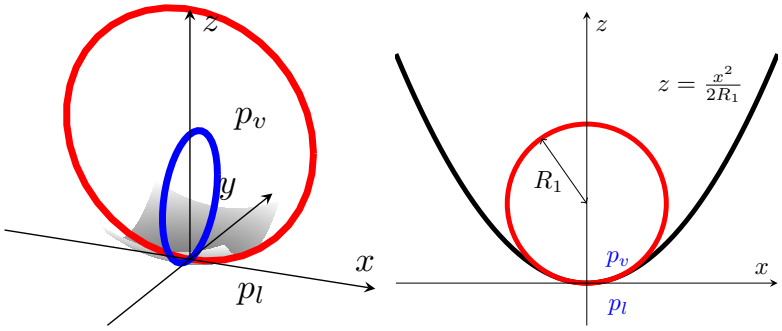

FIGURE 12. Curvature radii: Two- and one-dimensional illustrations.

surface can be parameterized by a function f as

$$z = f(x, y),$$

where O is the origin and f takes the form

$$f(x, y) = \frac{x^2}{2R_1} + \frac{y^2}{2R_2} + \text{lower order terms in } x \text{ and } y, \tag{13}$$

Let us consider a variation of the surface shape $f(\varepsilon, x, y)$, see Figure 13.

Question: ***Show that the pressure work when the surface moves, that is when ε spans [0,1], is equal to***

$$\delta W_p = (p_v - p_l)\delta V,$$

where V is the volume spans by the surface, see illustration in Figure 13. The work exterted by the surface tension is given by

$$\delta W_\sigma = \sigma \delta A$$

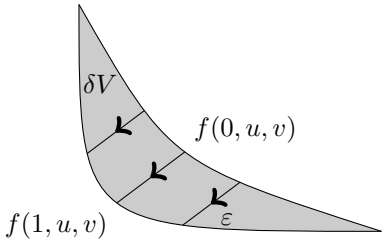

FIGURE 13. Deplacement of the surface. Here, u and v are the parameters for the surface given for a fixed ε. In Figure 12, they coincided with x and y. The volume δV spanned by the surface when ε goes from 0 to 1 is colored in gray

where δA denotes the change in the area of the surface. At equilibrium, we have

$$\delta W = \Delta p \delta V + \sigma \delta A = 0 \tag{14}$$

Question: ***Use the equilibrium condition*** (14) ***and the methodology of calculus of variation to obtain an equation for*** f

Question: ***Use the form of*** f ***given in*** (13) ***to derive the Young Laplace's equation.***

One dimensional model with mass exchange and capillary pressure

Let us now set up a model which combine the two fundamental aspects of a heat pipe: heat transfer using latent heat and capillary pressure drag. The model is depicted in Figure 14 at steady state. We have a closed loop with a vapor and liquid region. The values p_1^l, p_1^v, T_1 and T_2 denote the pressure and temperature of the liquid and vapor the interfaces, as indicated in Figure 14. The interface on the

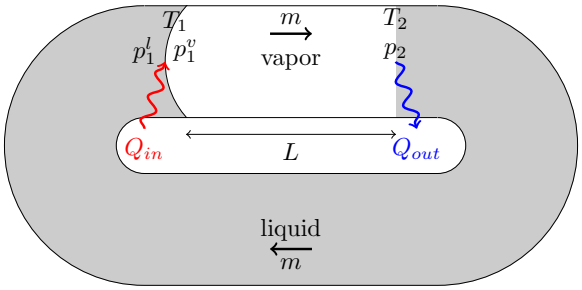

FIGURE 14. one-dimensional model with mass exchange at interfaces and capillary pressure jump at the liquid-vapor interfaces.

left has a curvature denoted Γ which enables a pressure jump. The pressure jump is given by the Young-Laplace equation,

$$p_1^v - p_1^l = \frac{2\sigma}{\Gamma}.$$

We consider one dimensional approximation of the flow and, as earlier, we simplify the momentum equation by taking Darcy approximations in both the liquid and vapor regions. We simplify further the model of the liquid region. We rewrite

the Darcy approximation as a linear dependency between mass rate and pressure gradient, that is,

$$m = -K_l(p_2^l - p_1^l),$$

for some given coefficient K_l. We neglect conduction and assume a steady state in the liquid so that the temperature gradient is constant and proportional to the temperature difference. On the interfaces, the energy balance give us the conditions

$$Q_{in} + \kappa_l(T_2 - T_1) = -\kappa_v \frac{\partial T}{\partial x}(0) + m h_{lv}, \tag{15a}$$

$$-\kappa_v \frac{\partial T}{\partial x}(L) + m_{lv} = Q_{out} + \kappa_l(T_2 - T_1). \tag{15b}$$

Here κ_l is a given constant.

Question: ***Explain the relations*** (15)**.**

We assume that the phases are at equilibrium so that

$$p_1^v = p_{\text{sat}}(T_1) \quad \text{and} \quad p_2 = p_{\text{sat}}(T_2).$$

The device will operate at a given ambient temperature T_{op}, so that $T_2 = T_{op}$, and given heat input Q_{in}.

Question: ***Compute the mass rate*** m ***and the heat released at the outlet*** Q_{out}**.**

Numerical simulation of a simplified 2D model including capillary effects

We add capillary effects to the numerical model developed previously. We include now the liquid region. The geometry is depicted in Figure 15. The boundary conditions are no flow on the walls, a constant heat flux Q_{in} on the left-hand side and a constant temperature T_0 on the right-hand side.

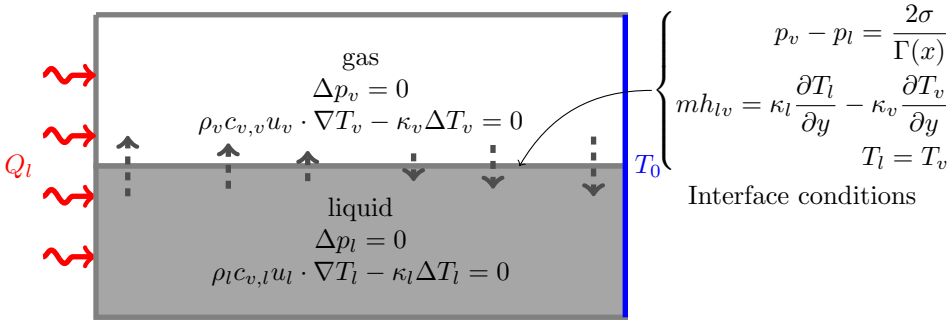

FIGURE 15. Numerical model including capillary effects.

Question: ***Explain the governing equations and interface conditions that are given in Figure 15***

Question: ***Solve numerically this system.***

References

[1] Wikipedia. URL: `https://en.wikipedia.org/wiki/Heat_pipe`.
[2] National Institute Of Standards and Technology. URL: `http://webbook.nist.gov/chemistry/fluid/`.

[3] Wikipedia. URL: https://en.wikipedia.org/wiki/Hagen-Poiseuille_equation.