Asymptotic Expansions

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Sinking object

Unscaled equation:

$$
\begin{aligned}
m\frac{d^2x^*}{dt^{*2}} &= m\tilde{g} - k\frac{dx^*}{dt^*},\\
x^*(0) &= 0,\\
\frac{dx^*}{dt^*}(0) &= V.
\end{aligned}
$$

Different rescalings possible dependent on the parameters and *the situation of interest*.

Sinking object, rescaled

	Equation	Parameters
large friction	$\varepsilon\ddot{x} + \dot{x} = 1$	$\varepsilon = 2\frac{V_\infty^2}{V_{FF}^2}$
long term	$\dot{x}(0) = \mu$	$\mu = \frac{V}{V_\infty}$
low friction	$2\ddot{x} + \varepsilon\dot{x} = 1$	$\varepsilon = \frac{V_{FF}}{V_\infty}$
long term	$\dot{x}(0) = \mu$	$\mu = \frac{V}{V_{FF}}$
large friction	$\ddot{x} + \dot{x} = \varepsilon$	$\varepsilon = \frac{V_\infty}{V}$
high velocity	$\dot{x} = 1$	

Case B — long term solution, low friction

Original ODE:

$$2\ddot{x} + \varepsilon\dot{x} = 1, \qquad x(0) = 0, \qquad \dot{x}(0) = \mu.$$

Note: by assumption $\varepsilon \ll 1$.

Approximate ODE by setting $\varepsilon = 0$:

$$2\ddot{x} = 1, \qquad x(0) = 0, \qquad \dot{x}(0) = \mu,$$

with solution

$$x_0(t) = \frac{t^2}{4} + \mu t.$$

- Is this an approximation of the actual solution with $\varepsilon \neq 0$?
- Can we find better approximations?

Perturbations of ODEs

Lemma

Consider the parameter dependent explicit ODE

$$
\begin{aligned}
x^{(k)}(t) &= F(t, x, \dot{x}, \ddot{x}, \ldots, x^{(k-1)}; \varepsilon), \\
x^{(s)}(0) &= y_s, \qquad s = 0, \ldots, k-1.
\end{aligned}
\tag{1}
$$

Assume that F is Lipschitz continuous and r-times continuously differentiable.
Denote by x_ε the solution of (1) for given ε. Then, for every t, the mapping

$$\varepsilon \mapsto x_\varepsilon(t)$$

is r-times continuously differentiable.

Asymptotic expansions — formal approach

Given a parameter dependent problem

$$\Phi(x;\varepsilon) = 0 \qquad \text{with } 0 < \varepsilon \ll 1.$$

1. Write the solution as

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$$

2. Write the problem as

$$0\Big(= \Phi(x;\varepsilon)\Big) = \Phi_0(x_0) + \varepsilon\Phi_1(x_0,x_1) + \varepsilon^2\Phi_2(x_0,x_1,x_2) + \cdots$$

3. Solve iteratively

$$\begin{aligned} \Phi_0(x_0) &= 0, && \text{solve for } x_0 \\ \Phi_1(x_0,x_1) &= 0, && \text{solve for } x_1 \\ \Phi_2(x_0,x_1,x_2) &= 0, && \text{solve for } x_2 \\ &\vdots \end{aligned}$$