The Method of Characteristics

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One-dimensional conservation laws

- Integral form:

$$\frac{d}{dt}\int_{x_0}^{x_1}\rho(x,t)\,dx + \big(j(t,x_1,\rho(x_1,t)) - j(t,x_0,\rho(x_0,t))\big) = \int_{x_0}^{x_1} q(t,x)\,dx.$$

- Differential form:

$$\frac{\partial}{\partial t}\rho(x,t) + \frac{d}{dx}j\big(t,x,\rho(x,t)\big) = q(t,x).$$

- Rewritten as:

$$\frac{\partial\rho}{\partial t} + a(t,x,\rho)\frac{\partial\rho}{\partial x} = b(t,x,\rho)$$

Method of characteristics

Choose $x_0 \in \mathbb{R}$ and consider the solution (the *characteristics* of the PDE)

$$x(t) =: X(t; x_0) \qquad\qquad z(t) =: Z(t; x_0)$$

of the system

$$\begin{aligned} \dot{x} &= a(t, x, z), & x(0) &= x_0, \\ \dot{z} &= b(t, x, z), & z(0) &= \rho_0(x_0), \end{aligned}$$

for some initial density ρ_0.
Then

$$\rho\big(X(t; x_0), t\big) = Z(t; x_0)$$

provided that nothing goes wrong.

Shocks

Shocks form when characteristics collide at some point $(\bar{x}, \bar{t})$.

Basic model:

- Up to the time $\bar{t}$, the density ρ is continuous.
- After the time $\bar{t}$, the density has a discontinuity (*shock*) along a curve $(s(t), t)$ starting at $(\bar{x}, \bar{t})$.
- Immediately to the left of the shock, the density and flux density are

$$\rho^-(t) := \rho(s(t)^-, t) \qquad j^-(t) := j(t, x, \rho(s(t)^-, t));$$

  immediately to the right, they are

$$\rho^+(t) := \rho(s(t)^+, t) \qquad j^+(t) := j(t, x, \rho(s(t)^+, t)).$$

- The shock develops at speed

$$\dot{s}(t) = \frac{j^+(t) - j^-(t)}{\rho^+(t) - \rho^-(t)} =: \frac{[j](t)}{[\rho](t)}.$$

Rarefaction waves

Rarefaction waves are formed, when a region of the (x, t) half-plane is not covered by characteristics.

Basic situation for equation of the form

$$\rho_t + j(\rho)_x = q(\rho) :$$

- We have a discontinuity at a point x_0 in the initial data.
- Characteristics starting near x_0 leave in opposite directions, creating a "dead sector" in between.
- Model the solution in the dead sector as

  $$\rho(x, t) = \varphi\Big(\frac{x - x_0}{t}\Big)$$

  such that the PDE holds.

Plan for the lecture

Traffic modelling with conservation laws:

- Develop a (very basic) PDE based model for the traffic density along a busy road.
- Study the behaviour of solutions for specific initial conditions:
  - Start of traffic flow at a traffic light.
  - Traffic behavious for increasing traffic density.
- Demonstrate that relatively simple models can be used to explain behaviour observed in reality.

Literature: Kompendium, pp. 184–194.