Conservation laws

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Basic ingredients

Quantities of interest:

- ρ . . . density.
- j . . . flux density.
- q . . . production rate (density).

Notation:

- R . . . control volume.
- ∂R . . . boundary of R.
- n . . . outward unit normal to R.

Conservation laws

- Integral form:
  For *each* control volume R and each time t we have

$$\frac{d}{dt}\int_R \rho(x,t)\,dV = -\int_{\partial R} j\big(x,t,\rho(x,t)\big)\cdot n\,d\sigma + \int_R q(x,t)\,dV.$$

  Change of quantity in $R = -$ flow through boundary $+$ production.

- Differential form:
  For each point (x,t) we have

$$\frac{\partial}{\partial t}\rho(x,t) + \nabla_x \cdot j\big(x,t,\rho(x,t)\big) = q(x,t).$$

One-dimensional situation

- Integral form:

$$\frac{d}{dt}\int_a^b \rho(x,t)\,dx + \big(j(b,t,\rho(b,t)) - j(a,t,\rho(a,t))\big) = \int_a^b q(x,t)\,dx.$$

- Differential form:

$$\frac{\partial}{\partial t}\rho(x,t) + \frac{d}{dx}j\big(x,t,\rho(x,t)\big) = q(x,t).$$

- Alternative formulation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial \rho}\frac{\partial \rho}{\partial x} = q(x,t) - \frac{\partial j}{\partial x}.$$

Plan for the lecture: Method of characteristics

- Method for solution of equations of the form

$$\frac{\partial u}{\partial t} + a(x,t,u)\frac{\partial u}{\partial x} = b(x,t,u)$$

  with initial values

$$u(x,0) = u_0(x).$$

- Idea: transformation into system of ODEs for each point x_0.
- Problems:
  - Discontinuities in the solution $\Rightarrow$ *shocks*.
  - Ambiguous solutions after shocks $\Rightarrow$ *rarefaction waves*.

Literature: Kompendium, pp. 177–184.