Dimensional analysis

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Trondheim,
August 29, 2018

Exercise classes

- Current favorite (at time of preparing the slides):
  - Wednesday 12–13 or 13–14.
- Try to move the exercises to one of these times (if available room can be found).
- Try to find a time slot for my office hours that particularly suits those that cannot attend the exercise class.
- Please fill out the doodle poll today.

Dimensional analysis

Physical relations can be rewritten as relations between dimensionless combinations.

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Swinging pendulum — relevant quantities:

- Frequency ω, dimension $[\omega] = 1/s$.
- Length L, dimension $[L] = m$.
- Mass m, dimension $[m] = kg$.
- Gravitational acceleration g, dimension $[g] = m/s^2$.
- Initial angle ϑ_0, dimension $[\vartheta_0] = 1$ (dimensionless!).

Dimensional analysis

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- Identify relevant physical quantities $R_1, \ldots, R_m$.
- Construct the dimension matrix $A \in \mathbb{R}^{n \times m}$ and its rank r.

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Swinging pendulum — dimension matrix:

	ω	L	m	g	ϑ_0
kg	0	0	1	0	0
m	0	1	0	1	0
s	-1	0	0	-2	0

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- Identify relevant physical quantities $R_1, \ldots, R_m$.
- Construct the dimension matrix $A \in \mathbb{R}^{n \times m}$ and its rank r.
- Select r core variables and find $m - r$ dimensionless combinations $\Pi_1, \ldots, \Pi_{m-r}$.

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Can choose core variables L, m, g, and obtain

$$\Pi_1 = \frac{\omega}{L^? m^? g^?} \stackrel{!}{=} \frac{\omega\, L^{1/2}}{g^{1/2}},$$
$$\Pi_2 = \frac{\vartheta_0}{L^? m^? g^?} \stackrel{!}{=} \vartheta_0.$$

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- Select r core variables and find $m - r$ dimensionless combinations $\Pi_1, \ldots, \Pi_{m-r}$.
- Deduce (Π-theorem) relation of the form

$$\Psi(\Pi_1, \ldots, \Pi_{m-r}) = 0.$$

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Swinging pendulum — defining relation:

$$\Psi(\Pi_1, \Pi_2) = \Psi\left(\omega\sqrt{\frac{L}{g}}, \vartheta_0\right) = 0.$$

- Argumentation shows that mass m of the pendulum is irrelevant.
- Physical argumentation: frequency depends uniquely on L, g, ϑ_0. Deduce therefore relation

$$\omega = \sqrt{\frac{g}{L}}\tilde{\Psi}(\vartheta_0).$$

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- Select r core variables and find $m - r$ dimensionless combinations $\Pi_1, \ldots, \Pi_{m-r}$.
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- Find Ψ if possible...

Dimensional analysis — model modifications

What happens if we add friction to the model?

- Cannot talk about frequency, as the motion won't be periodic. But still could say: ω ... inverse of time to the first local maximum.
- Friction is complicated — simplest model: friction force depends linearly on the velocity.
- Obtain friction force of the form

$$F_f = kv$$

  with:
  - v ... velocity of the (weight of the) pendulum.
  - k ... friction constant depending e.g. on medium, shape of the pendulum, ...
- Dimensions:

$$[k] = \frac{[F]}{[v]} = \frac{kg\ m}{s^2} \cdot \frac{s}{m} = \frac{kg}{s}.$$

Dimensional analysis — model modifications II

Obtain new (enlarged) dimension matrix:

	ω	L	m	g	ϑ_0	k
kg	0	0	1	0	0	1
m	0	1	0	1	0	0
s	-1	0	0	-2	0	-1

and dimensionless variables

$$\Pi_1 = \omega\sqrt{\frac{L}{g}}, \qquad \Pi_2 = \vartheta_0, \qquad \Pi_3 = \frac{k}{m}\sqrt{\frac{L}{g}}.$$

Obtain the relation:

$$\omega = \sqrt{\frac{g}{L}}\,\Psi\Big(\vartheta_0, \frac{k}{m}\sqrt{\frac{L}{g}}\Big).$$