Glacial Modelling

TMA 4195 — Mathematical Modelling
Modelling Project, Fall 2018

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Glaciers

Engabreen, Meløy kommune, Nordland.

Photo: Hallgeir Elvehøy, July 2015.

More glaciers

White Glacier, Nunavut, Canada.

Photo: Jürg Alean, July 2008.

Model equation

Can model the height h of a glacier by the equation

$$\frac{\partial}{\partial t} h + \lambda \frac{d}{dx} h^{m+2} = q.$$

Outline

Basic modelling

Goal:

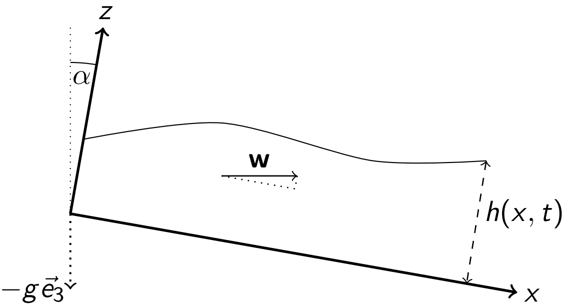
- Formulate equations that describe the dynamics that are going on in the interior of a glacier.

Main tools:

- Conservation laws for mass and momentum.

Mathematical set-up

- α ... slope of the valley.
- h ... height of the glacier.
- $\mathbf{w} = (u, v)$ ... velocity of the glacier.
- (x, z) ... coordinates along/above the valley.



Conservation laws — mass

- Mass is only removed on the surface $z = h(x,t)$ of the glacier.
- The ice is essentially incompressible.
- Can assume a constant density ρ of the glacial ice.

For each moving control volume $R = R(t)$ completely contained in the interior of the glacier we obtain

$$\frac{d}{dt} \int_{R(t)} \rho \, dx \, dz = 0$$

or in Eulerian, differential form

$$\nabla \cdot \mathbf{w} = \partial_x u + \partial_z v = 0.$$

(How is this equation obtained?)

Conservation laws — momentum

In Lagrangian form, we have the conservation of momentum (in a moving control volume $R(t)$)

$$\frac{d}{dt}\int_{R(t)} \rho \nabla \mathbf{w} = \int_{R(t)} f + \int_{\partial R(t)} T \cdot n$$

with

- f ... body forces (gravity).
- T ... stress, composed of pressure and viscous stresses, that is,

$$T = -p\,\mathrm{Id} + \tau \qquad \text{with} \qquad \tau = \begin{pmatrix} \tau_x \\ \tau_z \end{pmatrix} = \begin{pmatrix} \tau_{xx} & \tau_{xz} \\ \tau_{zx} & \tau_{zz} \end{pmatrix}.$$

We will ignore the term on the left hand side in the following.

(Why can we do that?)

Conservation laws — resulting equations

Decomposing the pressure as

$$p = \rho g \cos\alpha(h - z) + \tilde{p}$$

and inserting expressions for the force, we arrive at the equations (in Eulerian form)

$$\begin{aligned} \nabla \cdot \mathbf{w} &= 0, \\ \nabla \cdot \tau_x + \rho g \sin\alpha - \rho g \cos\alpha \partial_x h - \partial_x \tilde{p} &= 0, \\ \nabla \cdot \tau_z - \partial_z \tilde{p} &= 0. \end{aligned}$$

We still need a formulation for the (viscous) stress τ.

Stress tensor — Glen's law

A common assumption in ice-sheet modelling is *Glen's law*

$$e_{ij} = \mu\theta^{m-1}\tau_{ij}, \qquad i, j \in \{x, z\},$$

where

$$e_{ij} = \frac{1}{2}(\partial_i \mathbf{w}_j + \partial_j \mathbf{w}_i)$$

and

$$\theta^2 = \frac{1}{2}\sum_{i,j} \tau_{ij}^2.$$

Here μ is a (usually temperature dependent) constant and $m \approx 3$ some parameter.
Note that this implies that

$$\tau_{xz} = \tau_{zx} \qquad \text{and} \qquad \tau_{xx} = -\tau_{zz}.$$

(Why?)

Full set of PDEs

Collecting everything, we obtain the system of PDEs

$$
\begin{aligned}
\partial_x u + \partial_z v &= 0, \\
\partial_x \tau_{xx} + \partial_z \tau_{xz} + \rho g \sin\alpha - \rho g \cos\alpha \partial_x h - \partial_x \tilde{p} &= 0, \\
\partial_x \tau_{xz} - \partial_z \tau_{zz} - \partial_z \tilde{p} &= 0, \\
\partial_x u - \mu \theta^{m-1} \tau_{xx} &= 0, \\
\frac{1}{2}(\partial_z u + \partial_x v) - \mu \theta^{m-1} \tau_{xz} &= 0, \\
\theta^2 - \tau_{xx}^2 - \tau_{xz}^2 &= 0,
\end{aligned}
$$

for the functions u, v, $\tilde{p}$, τ_{xx}, τ_{xz}, θ in the interior of the glacier.
Note: We have a free boundary problem on the region

$$
\Omega(t) = \{(x,z) : 0 < z < h(x,t)\}.
$$

Mass conservation on the surface

Assume that ice is deposited/melts on the surface of the glacier at a rate $q(x,t)$. Conservation of matter then yields

$$\frac{d}{dt}\int_a^b h(x,t)\,dx + J(b,t) - J(a,t) = \int_a^b q(x,t),$$

where

$$J(x,t) = \int_0^{h(x,t)} u(x,z,t)\,dz$$

is the volume flux at x and t.

(Why?)

This can be used to derive a PDE that couples the height h of the glacier with the velocity u and the accumulation rate q.

(How?)

Outline

1. Ice-sheet modelling
2. Shallow ice approximation
3. Dynamics of glaciers

Approximation

Goal:

- Simplify the equations in the specific situation of a glacier model.
- Try to get rid of the free boundary problem.

Main tools:

- Dimensional analysis and rescaling.
- Asymptotic expansions.

Scaling

Typical situation in glaciers:

- Length L is much larger than (maximal) height H.
- E.g.: the tongue of Engabreen has a length of more than 1km, but a typical height of $\sim$50m.

We can rescale the variables as

$$x^* = Lx, \qquad z^* = Hz, \qquad h^* = Hh,$$

and then try to make use of the small parameter

$$\varepsilon = \frac{H}{L}.$$

- Scales for time, velocities, stresses, and pressure need to be based on balancing terms in the PDEs.

Asymptotic expansion

- Rescale the equations according to the scales chosen previously, and ignore all but the lowest order terms.
- We should obtain in particular the equations

$$\partial_z \tau_{xz} = \text{const} \qquad \text{and} \qquad \partial_z u = \text{const}\, |\tau_{xz}|^{m-1} \tau_{xz}.$$

- At the surface of the glacier, the shear stress τ_{xz} is equal to zero.
- If the glacier is frozen to its rock bed, then the velocity u at its bottom is equal to zero.
- This allows an explicit calculation of u (and v) only depending on h.

(Details?)

Resulting equation

With the asymptotic expansion chosen above, we arrive at the hyperbolic equation

$$\partial_t h + \lambda \frac{d}{dx} h^{m+2} = q.$$

Possible modifications:

- A non-uniform valley floor can be easily built into the equations. (How?)
- If the slope α of the valley is small as well, we have another small parameter in the set of equations. For the particular case $\alpha \sim \varepsilon$, a relatively simple modification of the same approach is possible, resulting in a non-linear, degenerate parabolic equation. (How?)

Outline

Numerical simulations

Goal:

- Use the model derived previously in order to simulate the dynamics of glaciers.

Main tools:

- Numerical methods for ODEs and PDEs.

Steady states

If the accumulation rate is (approximately) constant in time, it is possible that the glacier eventually reaches a steady state, where its flow perfectly balances the accumulation of ice at its top and the melting at its toe.

Questions:

- How many (if any) steady states are there? Can we compute them explicitly?
- Is it possible to say something about the stability of steady states?
- Will a glacier eventually reach its steady state?

Dynamics within a glacier

Even if a glacier appears stationary and its shape does not change over time, it is in permanent movement. Objects on the surface get covered in ice and will be slowly transported through the glacier, until they appear again, much further downstream, on its surface.

Questions:

- Compute (numerically) the trajectories of objects through a glacier.
- How long does it take until an object that is will appear again?

Dynamics of a glacier

Questions:

- Implement a numerical method for the solution of the (hyperbolic or degenerate parabolic) equation describing the movement of a glacier.
- Simulate with your method different scenarios like the formation of a glacier or changing climate conditions.