Singular perturbations

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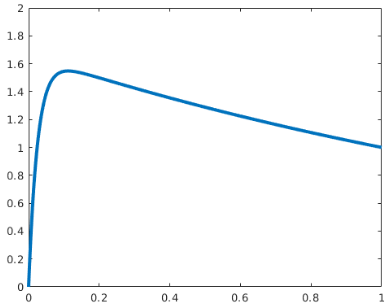
Singular perturbation for ODEs

Given the problem

$$\begin{aligned} \varepsilon y'' + 2y' + y &= 0, \\ y(0) &= 0, \\ y(1) &= 1. \end{aligned}$$

- Try to apply the ideas of asymptotic expansions in order to find reasonably good, simple approximations to the solution.
- Problem: for $\varepsilon = 0$, the character of the ODE changes.
- Simple application of asymptotic expansions cannot work because of conflicting boundary conditions.

Approximation with singular perturbations



Can clearly observe a *boundary layer* of width ε.

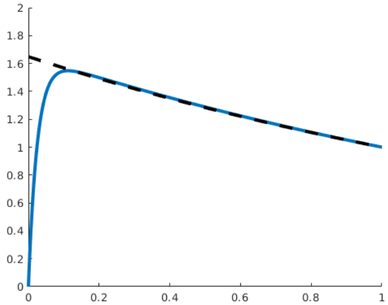
- Analytic solution:

$$y(x) = \frac{e^{r_+ x} - e^{r_- x}}{e^{r_+} - e^{r_-}}.$$

- Roots of the characteristic polynomial:

$$r_\pm = -\frac{1}{\varepsilon} \pm \sqrt{\frac{1}{\varepsilon^2} - \frac{1}{\varepsilon}}.$$

Approximation with singular perturbations



- Analytic solution:

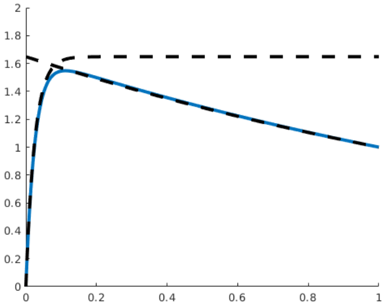
$$y(x) = \frac{e^{r_+ x} - e^{r_- x}}{e^{r_+} - e^{r_-}}.$$

- Outer solution:

$$y_O(x) = e^{\frac{1}{2} - \frac{x}{2}}.$$

- Provides a good approximation outside the boundary layer.

Approximation with singular perturbations



- Analytic solution:

$$y(x) = \frac{e^{r_+ x} - e^{r_- x}}{e^{r_+} - e^{r_-}}.$$

- Outer solution:

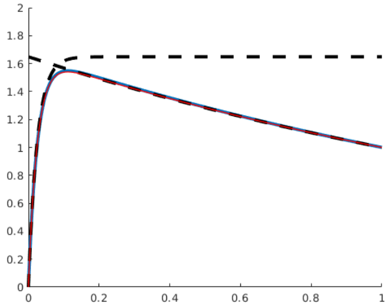
$$y_O(x) = e^{\frac{1}{2} - \frac{x}{2}}.$$

- Inner solution:

$$y_I(x) = e^{\frac{1}{2}}\left(1 - e^{-\frac{2x}{\varepsilon}}\right).$$

- Provides a good approximation inside the boundary layer.

Approximation with singular perturbations



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- Uniform approximation:

$$y_U(x) = e^{\frac{1}{2}}\left(e^{-\frac{x}{2}} - e^{-\frac{2x}{\varepsilon}}\right).$$