Stability of ODEs

Markus Grasmair

Department of Mathematics,
Norwegian University of Science and Technology,
Trondheim, Norway

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Equilibria

We consider a system of explicit, first order, autonomous ODEs

$$\dot{y} = F(y) \qquad \text{with} \qquad F\colon \mathbb{R}^n \to \mathbb{R}^n. \tag{1}$$

- An equilibrium point $y_e \in \mathbb{R}^n$ of (1) is a solution of the equation

$$F(y) = 0.$$

- In other words, y_e is an equilibrium point, if

$$y(t) = y_e$$

  is a solution of (1).

Stability of equilibria

- An equilibrium point y_e is stable if for every neighbourhood U of y_e there exists a neighbourhood V of y_e such that all solutions of (1) starting in V remain in U for all time.
- An equilibrium point y_e is asymptotically stable, if it is stable and there exists a neighbourhood U of y_e such that all solutions of (1) starting in U converge to y_e.
- An equilibrium point y_e is unstable, if it is not stable. That is: There exists a neighbourhood U of y_e such that for all neighbourhoods V of y_e there exists a solution of (1) starting in V and eventually leaving U.

Stability of linear systems

Consider specifically a linear system

$$\dot{y} = Ay \qquad \text{with } A \in \mathbb{R}^{n\times n} \text{ invertible.} \tag{2}$$

- The only equilibrium point of (2) is the point $y_e = 0 \in \mathbb{R}^n$.
- The point 0 is asymptotically stable if and only if

$$\operatorname{Re}\lambda_i < 0 \qquad \text{for every eigenvalue } \lambda_i \text{ of } A.$$

- The point 0 is unstable, if there exists an eigenvalue λ_i of A such that

$$\operatorname{Re}\lambda_i > 0.$$

- If the matrix A has n different eigenvalues and

$$\operatorname{Re}\lambda_i \leq 0$$

  for every eigenvalue λ_i of A, then 0 is stable.

Linearisation

Consider again a non-linear system

$$\dot{y} = F(y) \tag{3}$$

and an equilibrium point y_e of (3).

- The linearisation of (3) is the linear ODE

  $$\dot{\tilde{y}} = DF(y_e)\tilde{y}, \tag{4}$$

  where $DF \in \mathbb{R}^{n \times n}$ denotes the Jacobian of F.
- "Close to y_e, solutions of (3) look like functions of the form $y_e + \tilde{y}$, where $\tilde{y}$ solves (4)."[1]

[1] …provided that all eigenvalues of $DF(y_e)$ have a non-zero real part (*Hartman–Grobmann theorem*).

Stability of non-linear systems

Assume that y_e is an equilibrium point of (3) and denote by λ_i the eigenvalues of $DF(y_e)$.

- If $\operatorname{Re}\lambda_i < 0$ for all i, then y_e is asymptotically stable.
- If $\operatorname{Re}\lambda_i > 0$ for any i, then y_e is unstable.

Note: No information about stability in case $\operatorname{Re}\lambda_i \leq 0$ for all i if we do not have a strict inequality.

In particular case of a single ODE

$$\dot{y} = f(y) \qquad \text{with } f\colon \mathbb{R} \to \mathbb{R},$$

- y_e is asymptotically stable if $f_y(y_e) < 0$.
- y_e is unstable if $f_y(y_e) > 0$.