Introduction: Slides (general information)

## 1 Dimensional analysis (TMA4195, supplementary notes, chap 1)

Two basic facts about physical (and geometric) relations:

Come back to this
* All relations must be dimensionally correct.
* All relations are valid independent of choice of (fundamental) units.

A physical quantity R has

* dimension
* unit
* numerical value

Dimension: Length ($L$), Time ($T$), Mass ($M$), Energy ($ML^2T^{-2}$), ...

Unit: meter, feet, lightyear, months, Joule, Ampere, ...

$$R = 1m = 10^{-3}km = 39{,}37'' = 1{,}06 \cdot 10^{-15} \text{ lightyears}$$

In general: $R = \underset{\text{numerical value}}{V(R)} \underset{\text{unit}}{[R]}$

### Fundamental units

This is a set of units $F_1, \ldots, F_m$ such that any other unit can be written as $F_1^{a_1} F_2^{a_2} \ldots F_m^{a_m}$ for rational numbers $a_1, \ldots, a_m$, and such that $F_1^{b_1} \ldots F_m^{b_m} \neq 1$ for all $b_1, \ldots b_m$, except $b_1 = b_2 = \ldots = b_m = 0$.

## Example SI-system

| Dimension | Dimension symbol | SI unit |
|---|---|---|
| Mass | M | kilo (kg) |
| Length | L | meter (m) |
| Time | T | second (s) |
| Electric current | I | Ampere (A) |
| Absolute temperature | $\Theta$ | Kelvin (K) |
| Amount | N | Mole (mol) |
| Luminous intensity | J | Candela (cd) |

Slide of wikipedia definiton

## Back to the two basic facts:

* All relations must be dimensionally correct:

Forbidden: • $3m + 2kg = \ldots$

• $s = 4.9t^2$ (often seen in textbooks)

• $m = f(x,t)$ mass cannot be expressed by a relation involving only time and length!

$\ln(s)$, where $[s] = m$ is not possible

However, $E = mc^2$ is fine since $[E] = kg\,m^2 s^{-2}$ in SI-units

* All relations are valid independent of choice of fundamental units:

This means that if $f(R_1, \ldots, R_n) = 0$ is dimensionally correct, then

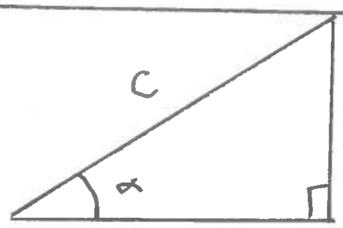
$$f(V(R_1), V(R_2), \ldots, V(R_n)) = 0 \quad \text{is true}$$

for any choice of fundamental units!

Thus, the function $f$ must have some type of inherent symmetry!

Dimensional analysis is a systematic procedure for exploiting this symmetry!

Example:




Right triangles

The area of a right triangle is uniquely defined by $c$ and $\alpha$. That is, there is a relation

$$A = f(c, \alpha)$$

giving the area $A$. Now if we rescale the triangle with a factor $a > 0$, we know the area is rescaled with the factor $a^2$. Thus

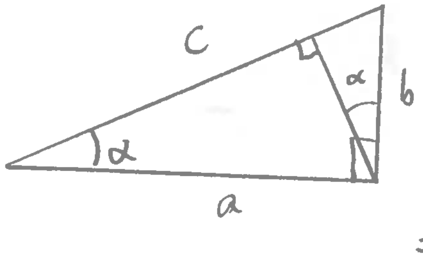
$$f(ac, \alpha) = a^2 f(c, \alpha)$$ (angle $\alpha$ does not change)

Set $c = 1$: $f(a, \alpha) = a^2 f(1, \alpha) = a^2 h(\alpha)$

for some function $h(\alpha)$. Thus, just by considering how length and area scales we obtain

$$A = c^2 h(\alpha)$$

Corollary: Pythagoras' theorem




From the figure we obtain by adding area

$$a^2 h(\alpha) + b^2 h(\alpha) = c^2 h(\alpha)$$

$$\Rightarrow a^2 + b^2 = c^2$$

Let us derive $A = c^2 h(\alpha)$ in a somewhat different manner:

We know there is some relation $g(A, c, \alpha) = 0$ (equivalent to $A = f(c,\alpha)$)

Let us make a <u>change of variables</u>

Put $\pi = \frac{A}{c^2}$ and keep variables $c$ and $\alpha$: Then

$$g(A, c, \alpha) = 0 \Leftrightarrow g(\pi c^2, c, \alpha) = 0 \Leftrightarrow m(\pi, c, \alpha) = 0$$

<u>Now comes the clue</u>: The relation $m(\pi, c, \alpha) = 0$ must be valid for any choice of length unit, i.e.

$$m(v(\pi), v(c), v(\alpha)) = 0$$

Now $\pi$ and $\alpha$ are dimensionless, they have no unit! Thus $\pi$ and $\alpha$ have always the same value independent of length unit, while $v(c)$ can be any (positive) number. The function $m$ is independent of $c$! We obtain

$$m(\pi, \alpha) = 0 \Leftrightarrow \pi = h(\alpha) \Leftrightarrow \underline{A = c^2 h(\alpha)}$$

<u>NB!</u> The last way of deriving $A = c^2 h(\alpha)$ is how we shall proceed in general in dimensional analysis.

<u>Example</u> <u>The first atomic explosion</u>

It might be plausible that the radius of the fireball of an atomic explosion depends roughly on

1. Time passed, $t$
2. Total energy of explosion, $E$
3. The density of the displaced medium (air), $\rho$

Thus we assume a relation

$$f(r,t,E,\rho) = 0 \qquad (*)$$

which must be valid for any choice of units for length, time, and mass. Again, let us try to make a new variable which is dimensionless. Using SI-units we have

$$[r] = m,\ [t] = s,\ [E] = kg\, m^2 s^{-2},\ [\rho] = kg\, m^{-3}$$

We observe that $\left[\frac{E}{\rho}\right] = m^2 s^{-2} (m^{-3})^{-1} = m^5 s^{-2}$

Thus $\pi = \frac{t^2 E}{\rho r^5}$ is dimensionless, and with this new variable instead of $E$, $(*)$ is equivalent to

$$h(\pi, r, t, \rho) = 0$$

valid for any choice of fundamental units. We realize that $v(r)$, $v(t)$, and $v(\rho)$ can be <u>any</u> numbers, thus $h$ is independent of $r, t$, and $\rho$ (!). We obtain

$$h(\pi) = 0 \Rightarrow \pi = C\ ,\ C \text{ constant and } [C] = 1.$$

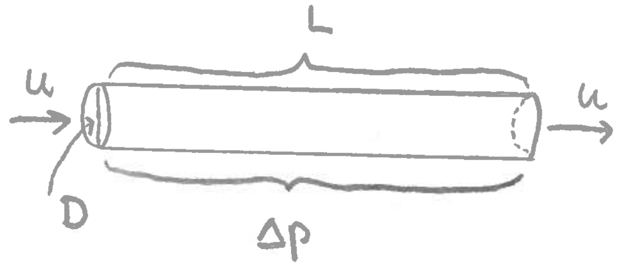
$$\pi = C \iff \frac{t^2 E}{\rho r^5} = C \iff \underline{E = C\rho r^5/t^2}\ ,\ C \text{ const.}$$

We have $\rho \approx 1.2\ kg/m^{-3}$, and from public pictures we have $r \approx 110\ m$ when $t = 16 \cdot 10^{-3}\ s$. Thus

$$E \approx C \frac{1.2 \cdot (110)^5}{16^2 \cdot 10^{-6}} kg\, m^2 s^{-2} = C \cdot 7{,}5 \cdot 10^{13}\ J$$

Assuming $C$ is of order 1, we have $\underline{E \approx 10^{14} J}$

Example: Steady state single-phase flow in a pipe




Length of pipe : $L$ , $[L] = m$
Diameter of pipe : $D$ , $[D] = m$
Flow velocity : $u$ , $[u] = m s^{-1}$
Pressure drop : $\Delta p$ , $[\Delta p] = kg\, m^{-1} s^{-2}$
Fluid viscosity : $\mu$ , $[\mu] = kg\, m^{-1} s^{-1}$
Fluid density : $\rho$ , $[\rho] = kg\, m^{-3}$
Pipe wall roughness : $E$ , $[E] = m$ ($E$ = "standard deviation of bumps")

We assume there is a relation

$$f(L, D, u, \Delta p, \mu, \rho, E) = 0 \qquad (*)$$

1. We start by choosing core variables. The core variables must be dimensionally independent; that is their numerical value can be given any value by changing fundamental units. (more on this topic after the example). So let $\rho, u, D$ be core variables (other choices also possible), and using the remaining 4 variables let us form 4 dimensionless new variables:

Let $\underline{\Pi_1 = \frac{L}{D}}$, $\underline{\Pi_2 = \frac{E}{D}}$ be the two first ones.

Now $[\Delta p] = kg\, m^{-1} s^{-2} \Rightarrow [\frac{\Delta p}{\rho}] = m^2 s^{-2}$, so then

$\underline{\Pi_3 = \frac{\Delta p}{\rho u^2}}$ is dimensionless. Finally $[\mu] = kg\, m^{-1} s^{-1}$,

so $[\frac{\mu}{\rho}] = m^2 s^{-1}$, thus $\frac{\mu}{\rho D u}$ is dimensionless. We will

define $\underline{\Pi_4 = \frac{\rho D u}{\mu}}$ as the last dimensionless variable.

Thus (*) is equvivalent to a dimensionally valid relation

$$g(\Pi_1, \Pi_2, \Pi_3, \Pi_4, \rho, u, D) = 0,$$

and since $v(\rho), v(u), v(D)$ can take any value, we must have

$$g(\Pi_1, \Pi_2, \Pi_3, \Pi_4) = 0, \text{ or}$$

$$\Pi_3 = h(\Pi_1, \Pi_2, \Pi_4).$$

Since we have steady state conditions, the pressure drop must be proportional to the pipe length. Thus

$$\Pi_3 = \Pi_1 k(\Pi_2, \Pi_4) \qquad (**)$$

for some function $k$. Writing out (**) we obtain

$$\Delta p = \frac{\rho u^2 L}{D} k\left(\frac{E}{D}, \frac{\rho u D}{\mu}\right) = \frac{\rho u^2 L}{D} k(e, Re)$$

where $e$ is the <u>relative roughness</u>, and $Re$ the famous <u>Reynolds number</u>

$f_F(Re, e) = \frac{1}{2} k(e, Re)$ is the <u>friction factor</u>

and can be determined from further analysis or measurements. That is, we need to cover a <u>2-dimensional</u> space $(Re, \varepsilon)$ of experiments to determine the relation $f(L, D, u, \Delta p, \mu, \rho, \varepsilon) = 0$ (!)

<u>Moral</u>: Dimensional analysis saves a lot of work for laboratory experimentalists!

See page 13 in compendium for more on $f_F$.

## <u>Buckingham's pi-teorem</u>

The last example prepares for the so-called Buckingham's pi-theorem in an excellent manner.

Assume we have a relation (dimensionally valid)

$$f(R_1, R_2, \ldots, R_n) = 0, \quad (*)$$

where $[R_j] = F_1^{a_{1j}} F_2^{a_{2j}} \ldots F_m^{a_{mj}}$, $F_1, \ldots F_m$ fundamental units. Then the <u>dimension matrix</u> is $m \times n$-matrix $A$

$$\begin{array}{c|cccc} & R_1 & R_2 & \ldots & R_n \\ \hline F_1 & a_{11} & a_{12} & \ldots & a_{1n} \\ F_2 & \vdots & \vdots & & \\ \vdots & & & & \\ F_m & a_{m1} & a_{m2} & \ldots & a_{mn} \end{array}$$

$A = \{a_{ij}\}_{i,j=1}^{m,n}$

We can now use well-known ideas from linear algebra to explain dimensional analysis!

In particular, the column-space of $A$; $col(A)$ explains "everything". rank $A$ is the dimension of $col(A)$.

1. $\mathrm{rank}(A)$ = number of dimensionally independent variables.
2. Write $A = [a_1, a_2, \ldots a_n]$, $\mathrm{rank}(A) = k$, ($a_i$ column vectors), Then $R_{i_1}, R_{i_2}, \ldots, R_{i_k}$ can be choosen as core variables if and only if $a_{i_1}, \ldots a_{i_k}$ is a basis for $col(A)$.
3. Then any other column vector ($n-k$ of them) is a linear combination of the basis vectors. This amounts to making $n-k$ dimensionless variables $\Pi_1, \ldots, \Pi_{n-k}$ using the $k$ core variables.
4. Thus we have

$$0 = f(R_1, \ldots, R_n) \Leftrightarrow 0 = g(\Pi_1, \ldots, \Pi_{n-k}, R_{i_1}, \ldots, R_{i_k}) = 0$$

$$\Leftrightarrow \quad g(\Pi_1, \ldots, \Pi_{n-k}) = 0$$ , since $V(R_{i_j})$ can be any value.

Example:

$$\begin{array}{c|cccc} & R_1 & R_2 & R_3 & R_4 \\ \hline F_1 & 1 & 2 & 1 & 1 \\ F_2 & -1 & 0 & 0 & 1 \\ F_3 & 0 & 0 & 1 & 0 \end{array} = A$$

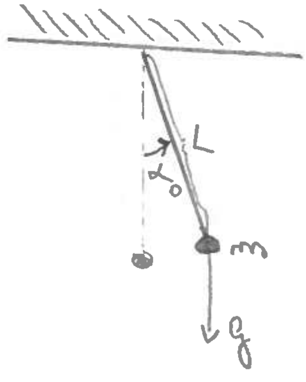
rank(A) = 3, and $a_3$ must be included in a basis for col(A). The possible basises for col(A) are $\{a_1, a_2, a_3\}$, $\{a_1, a_3, a_4\}$, and $\{a_2, a_3, a_4\}$.

Thus possible core variables are $\{R_1, R_2, R_3\}$, $\{R_1, R_3, R_4\}$, and $\{R_2, R_3, R_4\}$.

But, there is a catch here!

Here we can form $4-3=1$ dimensionless variable $\pi$, and since $R_3$ is the only variable containing $F_3$ in its unit, $\pi$ cannot be written as an expression containing $R_3$. Thus $f(R_1, .., R_4) = 0 \Leftrightarrow g(\pi) = 0$ means that $R_3$ cannot be included in $f(R_1, R_2, R_3, R_4) = 0$ in the first place (i.e. it is impossible to make a dimensionally valid expression $f(R_1, R_2, R_3, R_4) = 0$ where $R_3$ is actually included).

<u>Example</u> <u>Pendulum</u>




We want to find an expression for the frequency $\omega$ of a swinging pendulum, where we disregard air-resistance.

It is reasonable to assume

$$f(\omega, L, m, \alpha_0, g) = 0 \quad (*)$$

where $L$ is length of pendulum, $m$ the mass, $\alpha_0$ initial angle to equilibrium position, and $g$ the acceleration of gravity. We have dimension matrix:

$$A = \begin{array}{c|ccccc} & \omega & L & m & \alpha_0 & g \\ \hline m & 0 & 1 & 0 & 0 & 1 \\ s & -1 & 0 & 0 & 0 & -2 \\ kg & 0 & 0 & 1 & 0 & 0 \end{array}$$

We observe $\operatorname{rank}(A) = 3$, however, mass $m$ is the only variable with kg in its unit. Thus, mass cannot be included (*) if (*) is dimensionally correct!

Thus, dimension matrix is

$$A = \begin{array}{c|cccc} & \omega & L & \alpha_0 & g \\ \hline m & 0 & 1 & 0 & 1 \\ s & -1 & 0 & 0 & -2 \end{array}, \quad \operatorname{rank}(A) = 2.$$

Thus, choose $L$ and $g$ as core variables. $\alpha_0$ is already dimensionless, and we see $\pi = \frac{\omega}{\sqrt{g/L}}$ is dimensionless

Thus $f(\omega, L, \alpha_0, g) = 0 \Leftrightarrow g(\pi, \alpha_0, L, g) = 0$

$\Leftrightarrow g(\pi, \alpha_0) = 0 \Leftrightarrow \pi = h(\alpha_0) \Leftrightarrow \underline{\omega = \sqrt{g/L}\, h(\alpha_0)}$

Thus, fixing a planet (i.e. $g$) and a length $L$ we can measure $\pi = \frac{\omega}{\sqrt{g/L}} = h(\alpha_0)$ for varying values of $\alpha_0$ to estimate $h(\alpha_0)$ (again saving work in the lab!)

## Equivalent physics

If $f(R_1, \ldots, R_n) = 0 \iff g(\Pi_1, \ldots, \Pi_{n-k}) = 0$.

Then if two solutions to $f(R_1, \ldots, R_n) = 0$ $\{R_1, \ldots, R_n\}$ and $\{\hat{R}_1, \ldots, \hat{R}_n\}$ give rise to the same values for $\Pi_1, \ldots, \Pi_{n-k}$, then the two cases give "equivalent physics". This means that if you want to measure, say $R_1$, in terms of $R_2, \ldots, R_n$, you can instead consider a "model setup" with values $\hat{R}_2, \ldots, \hat{R}_n$, and measure $\hat{R}_1$.

For instance, in the exercises you are asked to demonstrate, using dimensional analysis, that ship design is difficult!

The values of the dimensionless numbers $\Pi_1, \ldots, \Pi_{n-k}$ in general define the physical regime for the problem at hand. E.g. for flow in pipe, the Reynolds number $R_e$ is strongly linked to how turbulent the flow is. Like $R_e < 2000$ means the flow is laminar (no turbulence). For much larger values of $R_e$ the flow is turbulent.