TMA4195, supplementary notes, chapter 2

2 Scaling

Introduction

Scaling is related to dimensional analysis, however, "the art of scaling" typically takes place in connection with a boundary-, or initial-value problem. That is, there is usually one (or several) differential equation(s) involved modelling a physical scenario.

Boundary value problems (BVP) are notoriously difficult to solve by hand in general. But even if this is so, one can often obtain insight to solutions of BVP by scaling the variables in the BVP in an appropriate manner. As indicated above, scaling is not an exact science, but the scaling exercise forces us to reflect on properties of the BVP, and quantify in a certain sense which effects are small, and which effects dominate the physics in a given BVP. Another conveinient feature of scaling is that it reduces the number of constants in a BVP, sometimes ridding the BVP for all constants.

The general feature of scaling is that we write a given BVP on dimensionless form by introducing "typical" scales for our variables. Defining, or obtaining (often by measurements), such scales can sometimes be quite straightforward, and other times it requires one to think hard about the qualitative nature of our physical problem.

Notation We need two names for each (non-dimensionless) variable, one for the standard physical variable that has units (e.g. SI-system), and one name for the corresponding dimensionless variable.

For a given physical variable u^* we estimate a "typical" size, or scale, for u^*, and call this scale U. Here $[u^*] = [U]$. Then define the variable u by

$$u^* = Uu,$$

meaning

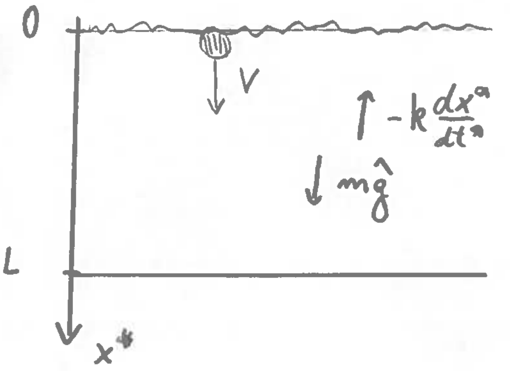
1. $[u] = 1$ (u is dimensionless)
2. The value of u is of order 1(*), or typically varies between 0 and 1 if $U = \max u^*$.

(*) of order 1 means that u is not very small ($\approx 10^{-2}$) or very large ($\approx 10^2$).

Case study: Sinking ball

In this example we consider a sinking ball in a liquid released with a given velocity. The mass density of the ball is assumed larger than the mass density of the liquid so that the ball does not float up to the surface

In addition to buoyancy and gravitation, we assume a friction force that is proportional to the velocity of the ball. (not very realistic though..)



Let t^* be time, $x^*(t^*)$ depth at time t^*, and let the "pool" have depth L.

The net force of gravity and buoyancy is $(V\rho_b - V\rho_l)g = V\rho_b\left(1 - \frac{\rho_l}{\rho_b}\right)g$

$= mg\left(1 - \rho_l/\rho_b\right)$, where

V is ball volume, $\rho_b > \rho_l$ are mass densities of ball and liquid, g the acceleration of gravity, and m mass of ball.

The friction force is $-k\frac{dx^*}{dt^*}$, where $[k] = \text{kg}\,\text{s}^{-1}$.

We assume $x^*(0) = 0$ and $\frac{dx^*}{dt^*} = v$. We then have the following initial value problem:

$$(*) \qquad m\frac{d^2x^*}{dt^{*2}} = -k\frac{dx^*}{dt^*} + m\hat{g}, \quad x^*(0) = 0, \quad \frac{dx^*}{dt^*}(0) = 0,$$

where $\hat{g} = g(1 - \rho_l/\rho_b) > 0$.

This initial value problem is in fact easy to solve exactly, but we will use this simple example to illustrate scaling tecniques applicable to also difficult problems, which are impossible, or hard, to solve by hand.

We can imagine some special cases for $(*)$:

If there is no friction ($k = 0\ \text{kgs}^{-1}$) and no initial velocity, we have free fall, and the ball hits the bottom $x^* = L$ with a velocity $\underline{V_F = \sqrt{2\hat{g}L}}$. (energy argument $\frac{1}{2} m V_F^2 = m\hat{g}L$)

On the other hand, if the friction is large, the ball will reach a constant velocity where (net) gravity and friction balances. Setting $\frac{d^2x^*}{dt^{*2}} = 0$ in $(*)$ we find

$$\underline{\frac{dx^*}{dt^*} = \frac{m\hat{g}}{k} = V_0}$$

(Note that if the initial velocity v is much large than v_0, the ball may not reach velocity v_0 before it hits bottom $x^* = L$)

We'll now first consider two cases (or physical regimes), where we will use v_0 and v_F to define natural timescales for the two cases:

<u>Case A, the friction is large, mainly constant velocity.</u>

If <u>not</u> $v \gg v_0$, and if k is "large", we can assume the ball falls with velocity v_0 "most of the way between $x^* = 0$ and $x^* = L$. Thus an approximate time scale T_0 is

$$T_0 = L/v_0 = \frac{Lk}{m\hat{g}},$$

and obviously a natural length scale is L.

We define dimensionless depth x and dimensionless time t by

$$x^* = Lx$$
$$t^* = T_0 t = \left(\frac{Lk}{m\hat{g}}\right) t$$

The functions $x^*(t^*)$ and $x(t)$ are related as

$$x^*(t^*) = L\,x\left(\frac{t^*}{T_0}\right). \qquad (**)$$

We see that x varies between 0 and 1, and that t varies from 0 to approximately 1.

Using $(**)$ we obtain the dimensionless form of $(*)$:

$$\frac{mL}{T_0^2}\frac{d^2x}{dt^2} = -\frac{kL}{T_0}\frac{dx}{dt} + m\hat{g} \iff$$

$$\frac{T_0 mL}{T_0^2 kL}\frac{d^2x}{dt^2} = -\frac{dx}{dt} + \frac{m\hat{g}T_0}{kL} \iff$$

$$\left(\frac{m^2\hat{g}}{Lk^2}\right)\frac{d^2x}{dt^2} = -\frac{dx}{dt} + 1 \iff \underline{\varepsilon\frac{d^2x}{dt^2} = -\frac{dx}{dt} + 1} \qquad (***)$$

where $\varepsilon = \frac{m^2\hat{g}}{Lk^2}$ is dimensionless (it must be!)

The initial values: $x^*(0) = 0 \Rightarrow \underline{x(0) = 0}$

$$\frac{dx^*}{dt^*}(0) = v \iff \frac{L}{T_0}\frac{dx}{dt}(0) = v \iff \underline{\frac{dx}{dt}(0) = \mu},$$

where $\underline{\mu = \frac{T_0 v}{L} = \frac{Lkv}{Lm\hat{g}} = \frac{kv}{m\hat{g}} = \frac{v}{v_0}}$

Such dimensionless numbers appearing in a scaling exercise tend to be ratios of physical quantities (obviously, both having the same dimension). We observe that μ is a ratio of two velocities v and v_0. Thus if $\mu > 1$ the motion slows down with time, and vice versa for $\mu < 1$. How about ε?

Observe that $\dfrac{v_0^2}{v_F^2} = \dfrac{m^2 \hat{g}^2}{k^2 2\hat{g}L} = \dfrac{m^2 \hat{g}}{2k^2 L}$. That is

$$\varepsilon = 2\frac{v_0^2}{v_F^2}$$, 2 times the ratio of kinetic energy at velocity v_0 and v_F. It is clear then (high friction $v_0 \ll v_F$) that $\varepsilon \ll 1$. That is <u>ε is small</u>.

Thus, we might want to find an approximation to the true solution by disregarding $\varepsilon \frac{d^2x}{dt^2}$ in (x**):

$$-\frac{dx}{dt} + 1 = 0 \quad , \; x(0) = 0 \, , \; x'(0) = \mu$$

giving(?) $x(t) = t$. However, unless $\mu = 1$, the problem has no solution, since $\frac{dx}{dt}(0) = \mu \neq 1$ cannot be met if we disregard the "small" term $\varepsilon \frac{d^2x}{dt^2}$. For $\mu \neq 1$ we imagine the change in velocity (i.e. $\frac{d^2x}{dt^2}$) is large at early times, so that $\varepsilon \frac{d^2x}{dt^2}$ is not small, and can not be disregarded in the beginning of the fall. At later times, it is perfectly appropriate to throw away $\varepsilon \frac{d^2x}{dt^2}$, then this term is actually small. In so-called singular perturbation theory (later in the course) we study these types of situations.

<u>Case B, small friction, approximately free fall</u>

When we have small friction, the falls almost freely.

A natural time scale now is now $T_F = L/V_F$, and the natural length scale is still L. Put

$$x^* = Lx \ , \quad t^* = T_F t = \frac{L}{V_F} t \ .$$

From (*) we obtain

$$\frac{Lm}{T_F^2} \frac{d^2x}{dt^2} = -\frac{kL}{T_F} \frac{dx}{dt} + m\hat{g}$$

$$\Leftrightarrow \frac{Lm}{(L/V_F)^2} \frac{d^2x}{dt^2} = -\frac{kL}{(L/V_F)} \frac{dx}{dt} + m\hat{g}$$

$$\Leftrightarrow \frac{V_F^2 m}{L} \frac{d^2x}{dt^2} = -V_F k \frac{dx}{dt} + m\hat{g}$$

$$\Leftrightarrow \left(\frac{V_F^2}{\hat{g}L}\right) \frac{d^2x}{dt^2} = -\frac{V_F k}{m\hat{g}} \frac{dx}{dt} + 1$$

$$\Leftrightarrow \lambda \frac{d^2x}{dt^2} = -\frac{V_F}{U_0} \frac{dx}{dt} + 1$$

$$\Leftrightarrow \underline{\lambda \frac{d^2x}{dt^2} = -\varepsilon \frac{dx}{dt} + 1} \ , \text{ where } \varepsilon = \frac{U_F}{U_0} \ .$$

$$\frac{dx^*}{dt^*}(0) = U \Leftrightarrow \frac{L}{T_F} \frac{dx}{dt}(0) = U \Leftrightarrow \underline{\frac{dx}{dt}(0) = \frac{T_F}{L} U = \frac{U}{U_F} = \mu}$$

Note that, if $\mu \gg 1$, i.e. $U \gg U_F$, $\frac{L}{U}$ would be a more appropriate timescale.

Assuming k is small, we have $U_0 \gg U_F$ and $\varepsilon \ll 1$.

Setting $\varepsilon = 0$ we obtain

$$\lambda \frac{d^2x}{dt^2} = 1 \ , \quad x(0) = 0, \quad \frac{dx}{dt}(0) = \mu \ ,$$ which is (of course) free fall.

Case c, high friction with $v \gg v_0$.

For this physical regime friction dominates over gravity, and let us find a natural timescale when the force is $-kv$. Then $m\dfrac{d^2x^*}{dt^{*2}} = -kv$, $x^*(0)=0$, $\dfrac{dx^*}{dt^*}(0) = v$

$\Rightarrow \dfrac{dx^*}{dt^*} = -\dfrac{kv}{m}t^* + v$. That is, the ball stops at $\underline{t^* = m/k = T}$

A length scale would be given by $\underline{L = vT = \dfrac{mv}{k}}$ (the ball actually travels $\frac{1}{2}L$, but using $\frac{1}{2}L$ or L as length scale actually is not important when scaling). Using

$$x^* = \frac{mv}{k}x \quad , \quad t^* = \frac{mv}{k}t \text{ , we obtain}$$

$$\frac{mL}{T^2}\frac{d^2x}{dt^2} = -\frac{kL}{T}\frac{dx}{dt} + m\hat{g} \quad \Leftrightarrow$$

$$\frac{d^2x}{dt^2} = -\frac{kT}{m}\frac{dx}{dt} + \frac{gT^2}{L} \quad \Leftrightarrow$$

$$\underline{\frac{d^2x}{dt^2} = -\frac{dx}{dt} + \varepsilon} \text{ , where}$$

$\underline{\varepsilon = \dfrac{m\hat{g}}{kv} = \dfrac{v_0}{v}}$, and $\dfrac{dx^*}{dt^*}(0) = v$ gives $\underline{\dfrac{dx}{dt}(0) = 1}$

Thus, setting $\varepsilon = 0$ gives

$$\frac{d^2x}{dt^2} = -\frac{dx}{dt} \quad , \; x(0) = 0 , \; \frac{dx}{dt}(0) = 1$$

with solution $\quad x(t) = 1 - e^{-t}$.

Summary and conclusions, sinking ball

We have considered a model for a sinking ball in some type of liquid. Three different physical regimes were studied:

A: $v_0 \ll v_F$, and not $v \gg v_0$ (high friction)

B: $v_0 \gg v_F$, (low friction and approximately free fall)

C: $v \gg v_0$, $v_F \gg v_0$ (v is dominating velocity)

Case	Dim.less param.	Scaled initial value prob.	Approximate initial value prob
A	$\varepsilon = \frac{v_0}{v_F}$, $\mu = \frac{v}{v_0}$	$2\varepsilon^2 x'' = -x' + 1$, $x(0) = 0$, $x'(0) = \mu$	$0 = -x' + 1$, $x(0) = 0$, $x'(0) = \mu$
B	$\varepsilon = \frac{v_F}{v_0}$, $\mu = \frac{v}{v_F}$	$2x'' = -\varepsilon x' + 1$, $x(0) = 0$, $x'(0) = \mu$	$2x'' = 1$, $x(0) = 0$, $x'(0) = \mu$
C	$\varepsilon = \frac{v_0}{v}$, $L_0 = \frac{L}{L_1}$, $L_1 = \frac{mv}{k}$	$x'' = -x' + \varepsilon$, $x(0) = 0$, $x'(0) = 1$	$x'' = -x'$, $x(0) = 0$, $x'(0) = 1$

Note that the approximate problem for case A has no solution except when $\mu = 1$, since the initial value $x'(0)$ is always 1 for the approximate equation. These matters will be considered when we investigate singular perturbation analysis.

Balancing the equations

Instead of taking a deep think of what typical scales are for different (extreme) physical regimes, we can proceed using a tecnique called balancing the equation. For the previous example the differential equation has three terms, where we found the ε changing from term to term.

according to which physical regime we were in.

The tecnique we now employ is the find time scale (and possibly length-scale) such that two of the terms in the equation have coefficients of order 1.

We have $m \dfrac{d^2 x^{*}}{dt^{*2}} = -k \dfrac{dx^{*}}{dt^{*}} + m\hat{g}$, $x^{*}(0) = 0$, $\dfrac{dx^{*}}{dt^{*}}(0) = v$

Let $x^{*} = Lx$ (L is a natural length scale), and let

$t^{*} = Tt$ for some time-scale.

This gives $\dfrac{mL}{T^2} x'' = -k \dfrac{L}{T} x' + m\hat{g}$, $x(0) = 0$, $\dfrac{L}{T} x'(0) = v$

Case A: Let's choose T such that the coefficient for x' is unity, and such that the constant term is unity.

We obtain: $\dfrac{m}{kT} x'' = -x' + \dfrac{m\hat{g}T}{kL}$

Thus if $\dfrac{m\hat{g}}{kL} T = 1$ we have balanced the equation

giving $T = \dfrac{kL}{m\hat{g}}$ (as previously), and we obtain

$$\left(\frac{m^2\hat{g}}{k^2 L}\right) x'' = -x + 1 \quad , x(0) = 0 \quad , x'(0) = v\frac{T}{L} = \frac{vk}{m\hat{g}}$$

This time, however, it is not obvious what the physical meaning of $\dfrac{m^2\hat{g}}{k^2 L}$ and $\dfrac{vk}{m\hat{g}}$ is not apperant.

Case B Coefficient of x'' unity, constant unity
(friction is small)

Again $x^\infty = Lx$, $t^\infty = Tt$ for some timescale T

From $\frac{mL}{T^2} x'' = -k \frac{L}{T} x' + m\hat{g}$ we obtain

$$x'' = -\frac{kT}{m} x' + \frac{\hat{g}T^2}{L}, \quad x(0)=0, \quad \frac{L}{T} x'(0) = v$$

We balance the equation by choosing T such that

$$\frac{\hat{g}T^2}{L} = 1 \Leftrightarrow \underline{T = \sqrt{L/\hat{g}}}, \text{ giving}$$

$$x'' = -\left(\frac{k}{m}\sqrt{L/\hat{g}}\right) x' + 1, \quad x(0)=0, \quad x'(0) = \frac{v}{\sqrt{L\hat{g}}}$$

Case C: In this case we choose both length-scale L_1 and timescale T_1 such that the coefficients for x'', $-x'$ are unity, and also such that $x'(0) = 1$.

From $\frac{dx^\infty}{dt^\infty}(0) = v \Leftrightarrow \frac{L_1}{T_1} x'(0) = v \Leftrightarrow x'(0) = v\, T_1/L_1 = 1$

$$\Rightarrow \frac{L_1}{T_1} = v \Leftrightarrow \underline{L_1 = vT_1}$$

From $\frac{mL_1}{T_1^2} x'' = -k \frac{L_1}{T_1} x' + m\hat{g}$

$$\Leftrightarrow \frac{mv}{T_1} x'' = -kv x' + m\hat{g}$$

$$\Leftrightarrow x'' = -\frac{kT_1}{m} x' + \frac{\hat{g}T_1}{v}$$

Thus $\frac{kT_1}{m} = 1 \Leftrightarrow \underline{T_1 = m/k}$, giving

$$x'' = -x' + \frac{\tilde{g} m}{kU}, \quad x(0) = 0, \; x'(0) = 1$$

The length scale $L_1 = UT_1 = \frac{mU}{k}$ and the depth L gives $\gamma = L/L_1$ which is the depth given in dimensionless length.

Some scaling considerations, turbulence

Fluids are mixed through <u>molecular diffusion</u> on the microscopic scale, and by convection on macroscopic scales.

Lets consider a whirl of diameter L and convection with velocity scale U. Then a timescale for this mechanism is $t_c = L/U$.

The timescale for diffusion is given by

$$t_d = L^2/\nu, \text{ where } \nu \text{ is kinematic viscosity.}$$

The Reynolds number is the ratio of these timescales

$$R_e = t_d/t_c = \frac{LU}{\nu}$$

<u>Example</u>: A $100\,m = L$ wide river, $U = 1\,ms^{-1}$, $\nu \approx 10^{-6}\,m^2 s^{-1}$

gives $R_e \approx \frac{100\,m \cdot 1\,ms^{-1}}{10^{-6}\,m^2 s^{-1}} = 10^8$

i.e. highly dominated by convection.

In turbulent flow, large whirls or vortices set smaller vortices moving and so on, until the vortices are so small that diffusion transforms the kinetic energy into thermal energy (friction). The loss of energy comes mainly from these small vortices with length scale l' and velocity u'. So if

K is energy loss per mass per time,

we expect $K = f(l', u', \nu)$; $[K] = m^2 s^{-3}$

From dimensional analysis we find

$$K = \nu\left(\frac{u'}{l'}\right)^2 g\left(\frac{l'u'}{\nu}\right) = \nu\left(\frac{u'}{l'}\right)^2 g(R_e)$$

When $t_c \approx t_d$, i.e. when $R_e = 1$ we have the smallest vortices, and we assume this kinetic energy is converted to heat energy.

Thus at this scale

$$K \approx \nu\left(\frac{u'}{l'}\right)^2 g(1),$$

and putting $g(1) = 1$ we have from

$$K = \nu\left(\frac{u'}{l'}\right)^2$$

and $R_e = 1 \Leftrightarrow \frac{l'u'}{\nu} = 1 \Leftrightarrow u' = \nu/l'$.

$$\Rightarrow K = \nu\left(\frac{\nu/l'}{l'}\right)^2 = \frac{\nu^3}{l'^4} \Leftrightarrow \underline{l' = \left(\frac{\nu^3}{K}\right)^{1/4}}$$

and $\underline{u' \approx (\nu K)^{1/4}}$

<u>These are Kolmogorov's microscales.</u>

Chapter 3 Regular perturbation

Regular perturbation is a method for obtaining an approximate solution to an equation (algebraic, transcendental, or differential) when the equation contains small parameter(s). The fundamental approach is to write the solution as a power series in terms of the small parameter(s).

Assume $0 < \varepsilon << 1$, and we have some equation for x (a number or a function $x(t)$) containing ε as a parameter:

$$D(x, \varepsilon) = 0$$

with solution $x(\varepsilon)$. If $\lim_{\varepsilon \to 0} x(\varepsilon) = x_0$, and if x_0 solves the equation $D(x, 0)$ we have a regular perturbation problem.

Example: $2\varepsilon^2 x'' = -x' + 1$, $x(0) = 0$, $x'(0) = \mu$

has a unique solution for $\varepsilon > 0$, but $0 = -x' + 1$, $x(0) = 0$, $x'(0) = \mu$ has no solution unless $\mu = 1$. Conclusion: this problem is not applicable for regular perturbation theory.

Example: $x^3 + x^2 + \varepsilon x - 2 = 0$, ε "small"

$\varepsilon = 0$ we have one real root $x = 1$.

Let $x = 1 + \varepsilon x_1 + \varepsilon^2 x_2 + O(\varepsilon^3)$. Insert into equation, ignoring $O(\varepsilon^3)$ terms:

$$(1 + \varepsilon x_1 + \varepsilon^2 x_2)^3 + (1 + \varepsilon x_1 + \varepsilon^2 x_2)^2 + \varepsilon(1 + \varepsilon x_1 + \varepsilon^2 x_2) - 2 = 0$$

$$\Leftrightarrow \left[1 + \varepsilon 3x_1 + \varepsilon^2(3x_1^2 + 3x_2)\right] + \left[1 + \varepsilon 2x_1 + \varepsilon^2(x_1^2 + 2x_1)\right]$$

$$+\left[\varepsilon(1 + \varepsilon x_1)\right] - 2 = 0 \quad \text{(ignoring } O(\varepsilon^3) \text{ terms)}$$

$$\Leftrightarrow \varepsilon(\underbrace{5x_1 + 1}_{=0}) + \varepsilon^2(\underbrace{4x_1^2 + 3x_2 + 3x_1}_{=0}) = 0$$

$$\Rightarrow x_1 = -1/5, \text{ and } \frac{4}{25} - \frac{3}{5} + 3x_2 = 0 \Leftrightarrow \underline{x_2 = \frac{11}{75}}$$

$$\underline{\text{Thus, } x \approx 1 - \frac{\varepsilon}{5} + \varepsilon^2 \frac{11}{75}}$$