TMA4195 Mathematical modelling, week 36, 2019

Regular perturbation theory, continuation

The projectile problem

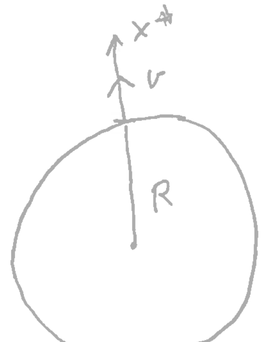
We'll consider a projectile beeing sent vertically upwards from the surface of the earth with a given initial velocity v which is much smaller than earth's escape velocity v_e. Since the gravitational acceleration is inversely proportional to the square of the distance to the centre of the earth, the initial value problem is

$$\frac{d^2x^*}{dt^{*2}} = -\frac{k}{(R+x^*)^2}, \quad x^*(0)=0, \quad \frac{dx^*}{dt^*}(0)=V.$$

Here R is the radius of earth, $x^*(t^*)$ the vertical height of the projectile above the surface of the earth at time t^*, and k some constant.

Since the acceleration is $-g$ for $x^*=0$ we find

$$k = R^2 g.$$



Thus,

$$\frac{d^2x^*}{dt^{*2}} = -\frac{gR^2}{(R+x^*)^2}, \quad x^*(0)=0, \quad \frac{dx^*}{dt^*}(0)=v.$$

This initial value can be integrated (solved) giving an explict exprsessin for $t^*(x^*)$ in terms of elementary functions. However, the inverse function, $x^*(t^*)$ cannot be expressed explicitly in terms of elementary functions.

Escape velocity

The energy required for overcoming the gravitational field, i.e. moving a mass m from $x^*=0$ to $x^*=\infty$, is given by

$$\int_0^\infty m \cdot \frac{R^2 g}{(R+x^*)^2}\, dx^* = mRg$$

So if $\frac{1}{2} m v^2 > mRg$ (v initial velocity), the projectile escapes. Thus $\underline{v_e = \sqrt{2Rg}} \approx 11.7\ \mathrm{km\,s^{-1}}$ is the escape velocity.

Scaling

We assume $v \ll v_e$, so that gravity does not decrease very much during the flight. Putting

$$\frac{d^2x^*}{dt^{*2}} = -g, \quad x^*(0) = 0, \quad \frac{dx^*}{dt^*}(0) = v$$

we find $\frac{dx^*}{dt^*}(t^*) = v - gt^*$.

Thus maximal height occurs at $t^* = v/g$

So, when $v << v_e$, $T = v/g$ is a reasonable timescale, and let $L = vT = v^2/g$ as time-scale. Thus, the dimensionless initial value problem becomes

$$\frac{L}{T^2}x'' = -\frac{R^2 g}{(R+Lx)^2}, \quad x(0)=0, \quad \frac{L}{T}x'(0) = v$$

where $x^* = Lx$ and $t^* = Tt$.

This gives $\quad x'' = -\dfrac{1}{(1+\varepsilon x)^2}, \quad x(0) = 0, \quad x'(0) = 1,$

where $\varepsilon = \dfrac{L}{R} = \dfrac{v^2}{Rg} = 2\dfrac{v^2}{(\sqrt{2Rg})^2} = \underline{2\dfrac{v^2}{v_e^2}}$

Thus, $\varepsilon << 1$ since we assume $v << v_e$.

Regular perturbation

For $|u| < 1$ we have $\dfrac{1}{1-u} = \sum_{n=0}^{\infty} u^n$ (geometric series).

Take the derivative on each side (termwise differentiation OK) giving $\dfrac{1}{(1-u)^2} = \sum_{n=1}^{\infty} n u^{n-1} = \sum_{n=0}^{\infty}(n+1)u^n$.

Thus $-\dfrac{1}{(1+\varepsilon x)^2} = -\dfrac{1}{(1-(-\varepsilon x))^2} = -\sum_{n=0}^{\infty}(n+1)(-\varepsilon x)^n$

$$= \sum_{n=0}^{\infty}(-1)^{n+1}(n+1)\varepsilon^n x^n = -1 + 2\varepsilon x - 3\varepsilon^2 x^2 + 4\varepsilon^3 x^3 + O(\varepsilon^4)$$

Put $x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \varepsilon^3 x_3(t) + O(\varepsilon^4)$

where $x_i(t)$, $i = 0, 1, \ldots$ are independent of ε.

We now insert this expression for $x(t)$ into

$$x'' = -\frac{1}{(1+\varepsilon x)^2} = -1 + 2x\varepsilon - 3x^2\varepsilon^2 + 4x^3\varepsilon^3 + O(\varepsilon^4),$$

where we disregard all $O(\varepsilon^4)$ terms.

Thus,

$$x_0'' + \varepsilon x_1'' + \varepsilon^2 x_2'' + \varepsilon^3 x_3'' = -1 + 2\varepsilon(x_0 + \varepsilon x_1 + \varepsilon^2 x_2)$$
$$-3\varepsilon^2(x_0+\varepsilon x_1)^2 + 4\varepsilon^3 x_0^3 + O(\varepsilon^4)$$
$$= -1 + 2\varepsilon x_0 + 2\varepsilon^2 x_1 + 2\varepsilon^3 x_2 - 3\varepsilon^2 x_0^2$$
$$- 6\varepsilon^3 x_0 x_1 + 4\varepsilon^3 x_0^3 + O(\varepsilon^4)$$
$$= -1 + \varepsilon 2x_0 + \varepsilon^2(2x_1 - 3x_0^2) + \varepsilon^3(2x_2 - 6x_0x_1 + 4x_0^3) + O(\varepsilon^4)$$

This gives

$$x_0'' = -1$$
$$x_1'' = 2x_0$$
$$x_2'' = 2x_1 - 3x_0^2$$
$$x_3'' = 2x_2 - 6x_0x_1 + 4x_0^3.$$

We have $x_n(0) = 0$ for $n = 0, 1, \dots$

$x_0'(0) = 1$, and $x_n'(0) = 0$ for $n = 1, 2, \dots$

Now the equations for $x_0, x_1, \dots$ are simple to solve, giving

$$\underline{x_0(t) = t - \tfrac{1}{2}t^2} \Rightarrow x_1'' = 2t - t^2$$

$$\Rightarrow \underline{x_1(t) = \tfrac{1}{3}t^3 - \tfrac{1}{12}t^4}$$

$$x_2'' = 2\left(\tfrac{1}{3}t^3 - \tfrac{1}{12}t^4\right) - 3\left(t - \tfrac{1}{2}t^2\right)^2$$

$$\Updownarrow$$

$$x_2'' = -3t^2 + \frac{11}{3}t^3 - \frac{11}{12}t^4$$

$$\Rightarrow x_2 = -\frac{1}{4}t^4 + \frac{11}{60}t^5 - \frac{11}{360}t^6 .$$

We stop here, and obtain

$$\underline{x(t) = t - \tfrac{1}{2}t^2 + \varepsilon\left(\tfrac{1}{3}t^3 - \tfrac{1}{12}t^4\right) + \varepsilon^2\left(-\tfrac{1}{4}t^4 + \tfrac{11}{60}t^5 - \tfrac{11}{360}t^6\right) + O(\varepsilon^3)}$$

So, what use is there in this expression? Let's estimate the time t_m it takes to reach maximal height.

Thus, we solve $x'(t) = 0 \Rightarrow$

$$1 - t + \varepsilon\left(t^2 - \tfrac{1}{3}t^3\right) + \varepsilon^2\left(-t^3 + \tfrac{11}{12}t^4 - \tfrac{11}{60}t^5\right) \approx 0$$

This is a fifth degree equation. But put

$$t = 1 + \varepsilon a + \varepsilon^2 b + O(\varepsilon^3),$$

and disregard $O(\varepsilon^3)$ terms:

$$1 - (1 + \varepsilon a + \varepsilon^2 b) + \varepsilon\left[(1+\varepsilon a)^2 - \tfrac{1}{3}(1+\varepsilon a)^3\right]$$
$$+ \varepsilon^2\left[-1 + \tfrac{11}{12} - \tfrac{11}{60}\right] = 0$$

$$\Leftrightarrow \varepsilon\left(\underbrace{-a + 1 - \tfrac{1}{3}}_{=0}\right) + \varepsilon^2\left(\underbrace{-b + 2a - a - \tfrac{4}{15}}_{=0}\right) = 0$$

Thus $a = 2/3$ and $b = \frac{2}{5}$,

giving $\underline{t_m = 1 + \frac{2}{3}\varepsilon + \frac{2}{5}\varepsilon^2 + O(\varepsilon^3)}$