Continues last week's lecture notes

Integrability of projectile problem

$$x'' = -\frac{1}{(1+\varepsilon x)^2} \Rightarrow x'' x' = -\frac{x'}{(1+\varepsilon x)^2}$$

$$\Leftrightarrow \frac{d}{dt}\left(\frac{1}{2}(x')^2\right) = \frac{d}{dt}\left(\frac{1}{\varepsilon}\frac{1}{(1+\varepsilon x)}\right)$$

$$\Rightarrow \frac{1}{2}(x')^2 = \frac{1}{\varepsilon}\frac{1}{1+\varepsilon x} + C$$

$x(0) = 0$, and $x'(0) = 1$, gives

$$(x')^2 = \frac{1-(2-\varepsilon)x}{1+\varepsilon x},$$

Setting $x'=0$, we obtain $x = \frac{1}{2-\varepsilon} = \frac{1}{2}\left(\frac{1}{1-\frac{1}{2}\varepsilon}\right)$

$= \frac{1}{2}\left(1+\frac{1}{2}\varepsilon+\frac{1}{4}\varepsilon^2\right) + O(\varepsilon^3)$, confirming the perturbation analysis.

We observe that for $\varepsilon > 2$, i.e. $v > v_e$, $1-(2-\varepsilon)x > 0$ for $x \geq 0$, meaning that $x' \neq 0$, which is to expect since $v > v_e$.

Now, assume $x' > 0$ (on our way up!), and $0 < \varepsilon < 2$.

Then $x' = \left(\frac{1-(2-\varepsilon)x}{1+\varepsilon x}\right)^{1/2} \iff \left(\frac{1+\varepsilon x}{1-\beta x}\right)^{1/2} dx = dt$

$x(0)=0$ gives

$$t(x) = \int_0^x \left(\frac{1+\varepsilon s}{1-\beta s}\right)^{1/2} ds,$$

where $\underline{\beta = 2-\varepsilon} > 0$. To calculate the integral, make the change of variable

$$v = \beta^{1/2}\left(\frac{1+\varepsilon s}{1-\beta s}\right)^{1/2} \quad \text{giving}$$

$\frac{(1-\beta s)^2}{\beta} v\,dv = ds$. Solve for s: $s = \frac{v^2-\beta}{\beta(v^2+\varepsilon)}$, gives

$$ds = \frac{1}{\beta} v \frac{4}{(v^2+\varepsilon)^2} dv,$$

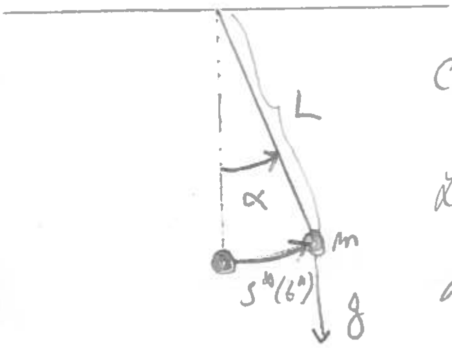
giving

$$t(x) = \int_0^x \left(\frac{1+\varepsilon s}{1-\beta s}\right)^{1/2} ds = \frac{4}{\beta^{3/2}} \int_{\beta^{1/2}}^{\beta^{1/2}\left(\frac{1+\varepsilon x}{1-\beta x}\right)^{1/2}} \frac{v^2}{(v^2+\varepsilon)^2}\, dv$$

Write $\frac{v^2}{(v^2+\varepsilon)^2}$ as $\underbrace{\frac{1}{2} v}_{s} \underbrace{\frac{2v}{(v^2+\varepsilon)^2}}_{r'}$, and use partial integration.

See Krogstad 2011 for more details.

Example Pendulum



Consider a pendulum as illustrated. Let $s^*(t^*) = L\alpha^*(t)$ be the distance, and the tangential force be $-gm \sin\alpha^*$. Newton's 2. law gives

$$mL\frac{d^2\alpha^*}{dt^{*2}} = -mg\sin\alpha^* \iff$$

$$\frac{d^2\alpha^*}{dt^{*2}} = -\frac{g}{L}\sin\alpha^*$$

We assume $\alpha^*(0) = \varepsilon$, $\frac{d\alpha^*}{dt^*}(0) = 0$, where ε is "small".

Scaling: Let $\alpha^* = \varepsilon\alpha$ and $t^* = Tt$ (for some T):

$$\frac{\varepsilon}{T^2}\alpha'' = -\frac{g}{L}\sin\varepsilon\alpha \iff \alpha'' = -\frac{1}{\varepsilon}\left(\frac{T^2 g}{L}\right)\sin\varepsilon\alpha$$

choose $T = \sqrt{L/g}$ as time-scale, giving

$$\alpha'' = -\frac{1}{\varepsilon}\sin \varepsilon\alpha \quad , \quad \alpha(0) = 1 \, , \, \alpha'(0) = 0$$

We have $\sin u = u - u^3/6 + O(u^5)$, so

$$\alpha'' = -\frac{1}{\varepsilon}\left(\varepsilon\alpha - \frac{\varepsilon^3}{6}\alpha^3 + O(\varepsilon^5)\right)$$

$$\Leftrightarrow \alpha'' = -\alpha + \frac{\varepsilon^2}{6}\alpha^3 + O(\varepsilon^4) \quad , \, x(0) = 1 \, , \, x'(0) = 0$$

<u>Perturbation</u>: Let $\alpha = \alpha_0 + \varepsilon\alpha_1 + \varepsilon^2\alpha_2 + O(\varepsilon^3)$, giving

$$\alpha_0'' + \varepsilon\alpha_1'' + \varepsilon^2\alpha_2'' = -\alpha_0 - \varepsilon\alpha_1 - \varepsilon^2\alpha_2 + \frac{\varepsilon^2}{6}\alpha_0^3 + O(\varepsilon^3).$$

Thus, $\alpha_0'' = -\alpha_0 \quad , \quad \alpha_0(0) = 1 \, , \, \alpha_0'(0) = 0$

$$\alpha_1'' = -\alpha_1 \quad , \quad \alpha_1(0) = \alpha_1'(0) = 0$$

$$\alpha_2'' = -\alpha_2 + \frac{\alpha_0^3}{6} \quad , \quad \alpha_2(0) = \alpha_2'(0) = 0 .$$

So $\alpha_0(t) = \cos t$ and $\alpha_1(t) = 0$. Thus

$$\alpha_2'' = -\alpha_2 + \frac{1}{6}\cos^3 t \quad , \quad \alpha_2(0) = \alpha_2'(0) = 0 .$$

We have $\cos^3 t = \left(\frac{1}{2}(e^{it} + e^{-it})\right)^3 = \frac{1}{8}\left(e^{3it} + e^{-3it} + 3(e^{it} + e^{-it})\right)$

$$= \frac{1}{4}(\cos 3t + 3\cos t)$$

Thus a particular solution is of the form

$$\alpha_{2p} = A\cos 3t + B\sin 3t + Ct\cos t + Dt\sin t \quad \text{(ressonance)}$$

giving $\alpha_2 = \alpha_{2H} + \alpha_{2p} = c_1\cos t + c_2\sin t - \frac{1}{192}\cos 3t + \frac{1}{16}t\sin t$

$\alpha_2(0) = \alpha_2'(0) = 0$ gives $\alpha_2(t) = \frac{1}{192}(\cos t - \cos 3t) + \frac{1}{16}t\sin t$

Thus,

$$\alpha(t) = \cos t + \varepsilon^2\left(\frac{1}{192}(\cos t - \cos 3t) + \frac{1}{16} t \sin t\right) + O(\varepsilon^3)$$

$\alpha(t)$ should be <u>periodic</u>, however the term $t \sin t$ is non-periodic. This illustrates that the pertubation "solution" has a qualitative different property than the exact solution. A more elaborate approach is the <u>Poincare - Linstedt</u> method. We write

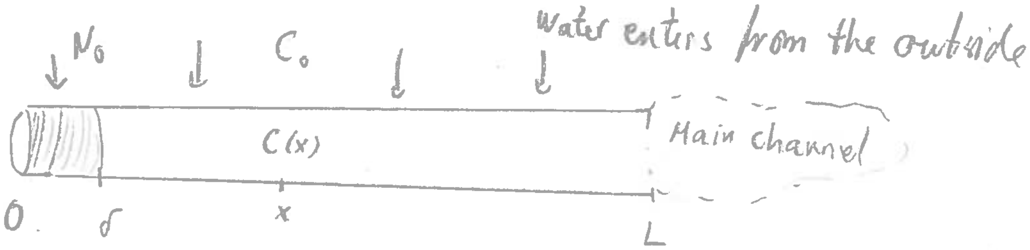
$$\alpha(t) = \alpha_0(w_\varepsilon t) + \varepsilon \alpha_1(w_\varepsilon t) + \varepsilon^2 \alpha_2(w_\varepsilon t) + O(\varepsilon^3),$$

where
$$w_\varepsilon = 1 + \varepsilon w_1 + \varepsilon^2 w_2 + \dots,$$
and then try to find $w_1, w_2, \dots$ and $\alpha_0, \alpha_1, \dots$ such that $\alpha_0, \alpha_1, \dots$ are periodic. <u>See exercise</u>!

<u>Modelling the kidney function</u>

We'll now model the flow of water and salt ions in so-called secondary channels to the main channels in the kidney. We will assume the salt ion concentration in the main channel and in the surrounding tissue to the secondary channels (SC) is constant, and equal to C_0. At the start of a SC there is a chemical pump, where salt ions enter at a rate N_0, where

$[N_0] = \text{osmol}/m^2 s$ (number of ions (in mole) per area per time)



The "start" of a SC is the interval $[0,\delta]$, thus the osmol of ions entering a SC per second is $N_0 \delta c$, where c is the circumference to the SC.

Water enters a SC from the surrounding tissue with a flux

$$J = P(C(x) - C_0)$$

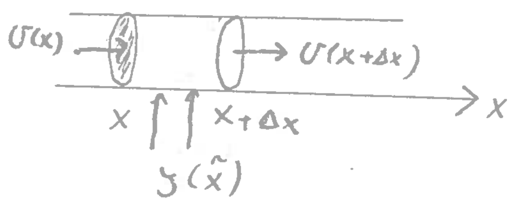
through the mechanism of osmosis. Here $[C]=$

Note: We will avoid using the x^{∞}-notation until the full boundary value problem is derived.

Here P is membrane permeability, $[P] = \text{osmol}^{-1} m^4 s^{-1}$, and $C(x)$ is the osmol concentration of salt ions, $[C] = \text{osmol} \cdot m^{-3}$

Water, mass concervation

Let $v(x)$ be water velocity at position x, and let A be the cross-sectional area of the SC.



Assuming the water is incompressible, we must have

$$v(x)A + J(\hat{x})c\Delta x = v(x+\Delta x)A \text{, for some } \hat{x} \in [x, x+\Delta x].$$

Thus $J(\tilde{x})\frac{C}{A} - \left(\frac{v(x+\Delta x)-v(x)}{\Delta x}\right) = 0$. Letting $\Delta x \to 0$ we have

$$J(x)\frac{C}{A} - v'(x) = 0 \iff \boxed{\left(\frac{P_t}{A}\right)(c(x)-c_0) - v'(x) = 0}$$

stating mass-conservation for water (or volume concentration since water is incompressible)

Mass conservation for salt ions

At $x = L$, where the SC meets the main channel, we'll assume $c(L) = c_0$. Transport of salt ions in a SC is governed by two mechanisms,

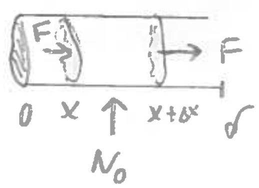
convective flow : $c(x)v(x)$
molecular diffusion : $-D\,c'(x)$ (Fick's law)

The flux of salt ions in the SC is then

$$F(x) = c(x)v(x) - D\,c'(x),$$

where $[F] = \text{Osmol}\, m^{-2}s^{-1}$.

Now, for $x \in \langle 0, \sigma \rangle$ salt ions enters into the SC.



Assume $0 < x < x+\Delta x < \sigma$. Then

$$F(x)A + N_0 C\,\Delta x = F(x+\Delta x)A$$

$$\iff \frac{F(x+\Delta x)-F(x)}{\Delta x} = \frac{N_0 C}{A}, \text{ and } \Delta x \to 0$$

gives

$$F'(x) = \frac{N_0 C}{A} \quad \text{for } x \in \langle 0, \sigma \rangle.$$

For $x > \sigma$ we have $N_0 = 0$, giving

$$F'(x) = 0 \quad \text{for } x \in \langle \sigma, L \rangle.$$

Since the end wall at $x=0$ is impermeable (no flow) we have $F(0)=0$. Thus we obtain

$$F(x) = \begin{cases} \frac{N_0 c}{A} x & \text{for } x \in [0,\delta] \\ \frac{N_0 c}{A} & \text{for } x \in \langle \delta, L] \end{cases}$$

i.e. $F(x)$ is a known function (given by N_0, S, A, and δ).

<u>Boundary values.</u>

We already have $\underline{c(L)=C_0}$, and since there is no flow of water at $x=0$ (no water flows in from the start at $x=0$) we have $\underline{v(0)=0}$. Since $F(0)=0$, we have automatically

$$F(0) = c(0)v(0) - D c'(0) = 0 \Rightarrow \underline{c'(0)=0}$$

Finally, we must ensure that <u>c and v are continuous</u> at $x=\delta$. Thus, we have arrived at the following boundary value problem for $c^*(x^*)$ and $v^*(x^*)$ (reintroduce the star notation)

$$\left.\begin{aligned} \frac{dv^*}{dx^*} &= \left(\frac{Pc}{A}\right)\left(c^*(x^*) - C_0\right) \\ v^*(x^*)c^*(x^*) - D\frac{dc^*}{dx^*}(x^*) &= F^*(x^*) \end{aligned}\right\} \text{ for } x^* \in [0,L]$$

$c^*(L)=0$, $v^*(0)=0$, $\left(\frac{dc^*}{dx^*}(0)=0 \text{ follows automatically}\right)$

c, v continuous, where

$$F^*(x^*) = \begin{cases} \frac{N_0 c}{A} x^* & \text{for } x^* \in [0,\delta] \\ \frac{N_0 c}{A} & \text{for } x^* \in \langle \delta, L] \end{cases}.$$

Scaling

A natural scale for ion concentration c^* is C_0. For length we have two natural scales, either δ or L. Which one we chose is not important, and we let δ be our length scale. Velocity might not be so obvious. Now, we know that salt is injected with a rate $N_0 c\delta$. If the same amount should leave the SC (at $x^* = L$) through the process of convection alone with a velocity V, we must have (remember $c^*(L^*) = C_0$)

$$V C_0 A = N_0 c \delta \Leftrightarrow \boxed{V = \frac{N_0 c \delta}{C_0 A}}$$

(the actual velocity $v^*(L)$ will be smaller since some of the ions are transported by diffusion).

Thus,

$$x^* = \delta x$$

$$c^* = C_0 c$$

$$v^* = V v$$

giving

$$\frac{V}{\delta} v'(x) = \frac{P_c C_0}{A} (c(x) - 1)$$

$$V C_0 v(x) c(x) - \frac{D C_0}{\delta} c'(x) = \begin{cases} \frac{N_0 c \delta}{A} x & 0 \le x < 1 \\ \frac{N_0 c \delta}{A} & 1 \le x \le \lambda \end{cases}$$

$v(0) = 0$, $c(\lambda) = 1$, c & v continuous.

Here $\boxed{\lambda = L/\delta}$ is dimensionless length of the SC.

This gives

$$\left.\begin{array}{ll}(1) & \varepsilon v' = c - 1 \\ (2) & vc - \eta c' = \binom{x}{1}\end{array}\right\} \quad 0 \leq x \leq \lambda$$

"Dimensionless groups"

$v(0) = 0$, $c(\lambda) = 1$, v and c continuous.

$$\boxed{\varepsilon = \frac{N_0}{PC_0^2}, \quad \eta = \frac{AC_0D}{N_0\delta^2 c} \qquad \lambda = L/\delta}$$

Here $\binom{x}{1} = \begin{cases} x & \text{for } 0 \leq x < 1 \\ 1 & \text{for } x \geq 1 \end{cases}$

Regular perturbation

Measurements indicate, or estimate, the range in value of the different variables, $N_0, L, \delta, P, c, D, C_0$. This indicates that $\varepsilon << 1$, and we use ε for our perturbation variable. Let's drop all $O(\varepsilon^2)$-terms:

$$c(x) = c_0(x) + \varepsilon c_1(x) + O(\varepsilon^2)$$
$$v(x) = v_0(x) + \varepsilon v_1(x) + O(\varepsilon^2) \quad \Rightarrow$$

(1) $\Rightarrow$ $\varepsilon v_0' = c_0 + \varepsilon c_1 - 1 + O(\varepsilon^2)$

$\Leftrightarrow$ $c_0 - 1 + \varepsilon(c_1 - v_0') = O(\varepsilon^2)$,

that is: $\boxed{\begin{array}{l} c_0 = 1, \\ c_1 = v_0' \end{array}}$

using $c_0 = 1$

(2) $\Rightarrow$ $(v_0 + \varepsilon v_1)(1 + \varepsilon c_1) - \eta(\varepsilon c_1') = \binom{x}{1} + O(\varepsilon^2)$

$\Leftrightarrow$ $\left(v_0 - \binom{x}{1}\right) + \varepsilon(v_0 c_1 + v_1 - \eta c_1') = O(\varepsilon^2)$

Thus $v_0 = \binom{x}{1}$ and $c_1 = v_0' = \binom{1}{0} = \begin{cases} 1 & \text{for } 0 \leq x < 1 \\ 0 & \text{for } x > 1 \end{cases}$

So $c_1(x)$ is discontinuous, and this is unphysical! (would make $c(x)$ discontinuous). Something is not as they should be. From measurements it is seen that N_0 can vary between 10^{-10} mOsmol cm^{-2} s^{-1} and 10^{-5} mOsmol cm^{-2} s^{-1} (the first "m" is milli), and the smaller ε gets, the larger η typically gets. So

$$v(x)\,c(x) - \eta c'(x) = \binom{x}{1}$$

has 2 terms ≈ 1 and η is typically large. The right approach is to interpret this equation as having one term ≈ 1 and two small terms.

New perturbation approach

We multiply $vc - \eta c' = \binom{x}{1}$ by ε to obtain

$$\varepsilon v c - (\varepsilon\eta) c' = \varepsilon \binom{x}{1},$$

where $\varepsilon\eta$ does not contain N_0, thus rendering $\varepsilon\eta$ as a more stable coefficient.

Actually it is convenient for notational purposes to write

$$\varepsilon k^2 v c - (\varepsilon\eta) k^2 c' = \varepsilon k^2 \binom{x}{1},$$

where k is defined by $\varepsilon\eta k^2 = \lambda^2 \Leftrightarrow \boxed{k^2 = \lambda^2/\varepsilon\eta}$

Thus we arrive at

$$\left.\begin{array}{c} \varepsilon v' = c - 1 \\ \varepsilon k^2 v c - \lambda^2 c' = \varepsilon k^2 \binom{x}{1} \\ v(0)=0,\ c(\lambda) = 1,\ v \,\&\, c \text{ continuous.} \end{array}\right\}$$

This introduction of ε in the second equation changes the perturbation analysis, and allows us to find physical sensible answers. So let

$$C(x) = C_0(x) + C_1(x)\varepsilon + O(\varepsilon^2)$$
$$v(x) = v_0(x) + v_1(x)\varepsilon + O(\varepsilon^2)$$

We will find c_0, C_1, and v_0: Inserted into the equations:

$$\begin{cases} \varepsilon(v_0') = C_0 + \varepsilon C_1 - 1 + O(\varepsilon^2) \\ \varepsilon k^2 v_0 C_0 - \lambda^2 (C_0' + \varepsilon C_1') = \varepsilon k^2 \binom{x}{1} + O(\varepsilon^2) \end{cases}$$

$$\hookrightarrow \begin{cases} C_0 - 1 + \varepsilon(C_1 - v_0') = O(\varepsilon^2) \\ -\lambda^2 C_0' + \varepsilon\left(k^2 v_0 C_0 - \lambda^2 C_1' - k^2\binom{x}{1}\right) = O(\varepsilon^2) \end{cases}$$

Giving:
$$C_0 = 1$$
$$C_1 = v_0'$$
$$k^2 v_0 C_0 - \lambda^2 C_1' = k^2 \binom{x}{1}.$$

We have $C_1' = v_0''$, thus the last equation is

$$\boxed{k^2 v_0 - \lambda^2 v_0'' = k^2\binom{x}{1}}$$

We have $v(0) = 0 \Rightarrow v_0(0) = 0$. Using $C(\lambda) = 1$ gives

$$C(\lambda) = 1 + \varepsilon C_1(\lambda) + O(\varepsilon^2) = 1 \Rightarrow C_1(\lambda) = v_0'(\lambda) = 0$$

In addition, $C_1(x)$ and $v_0(x)$ must be continuous, i.e. $v_0(x)$ and $v_0'(x)$ must be continuous.

These 4 conditions ($v_0(0)=0$, $v_0'(\lambda)=0$, v_0 and v_0' cont.) (18)
will determine 4 integration constants appearing below.

We see that for $0 \le x < 1$ v_0 satisfies

$$-\lambda^2 v_0'' + k^2 v_0 = k^2 x ,$$

which has general solution

$$v_0(x) = x + A_1 \cosh(\varphi x) + B_1 \sinh(\varphi x) ,$$

where $\boxed{\varphi = k/\lambda}$, A_1 and B_1 integration constants.

$v_0(0) = 0 \Rightarrow \underline{v_0(x) = x + B_1 \sinh \varphi x}$, $\underline{0 \le x < 1}$

For $1 \le x \le \lambda$:

$$-\lambda^2 v_0'' + k^2 v_0 = k^2$$

$$\Rightarrow \quad v_0(x) = 1 + A_2 \cosh \varphi x + B_2 \sinh \varphi x$$

$$\Rightarrow \quad v_0'(x) = A_2 \varphi \sinh \varphi x + B_2 \varphi \cosh \varphi x$$

$C_1(\lambda) = v_0'(\lambda) = 0$ gives

$$A_2 \varphi \sinh \varphi \lambda + B_2 \varphi \cosh \varphi \lambda = 0 \quad (\text{note } \varphi\lambda = k)$$

$$\Rightarrow \quad B_2 = -A_2 \frac{\sinh k}{\cosh k} .$$

Thus
$$v_0(x) = 1 + A_2 \left(\cosh \varphi x - \frac{\sinh k}{\cosh k} \sinh \varphi x \right)$$

$$= 1 + \frac{A_2}{\cosh k} \left(\cosh \varphi x \cosh k - \sinh \varphi x \sinh k \right)$$

$$= 1 + \hat{A}_2 \cosh(\varphi x - k) \quad , \text{ where}$$

$\hat{A}_2 = \frac{A_2}{\cosh k}$ is arbitrary new constant.

Thus:

$$U_0(x) = \begin{cases} x + B_1 \sinh \varphi x & 0 \leq x < 1 \\ 1 + \hat{A}_2 \cosh(\varphi x - k) & 1 \leq x \leq \lambda \end{cases}$$

Now, require continuity at $x=1$ for U_0 and U_0':

We have $U_0(1^-) = U_0(1^+) \Leftrightarrow$

$$1 + B_1 \sinh \varphi = 1 + \hat{A}_2 \cosh(\varphi - k)$$

$$\Leftrightarrow \quad \underline{B_1 \sinh \varphi - \hat{A}_2 \cosh(\varphi - k) = 0} \qquad (1)$$

$U_0'(1^-) = U_0'(1^+)$ gives

$$1 + B_1 \varphi \cosh \varphi = \hat{A}_2 \varphi \sinh(\varphi - k) \Leftrightarrow$$

$$\underline{-B_1 \varphi \cosh \varphi + \hat{A}_2 \varphi \sinh(\varphi - k) = 1} \qquad (2)$$

(1) and (2) then determines B_1 and $\hat{A}_2$:

$$B_1 = -\frac{\cosh(\varphi - k)}{\varphi \cosh k} \quad , \quad \hat{A}_2 = -\frac{\sinh \varphi}{\varphi \cosh k}$$

(Note! Got this wrong in the lecture)

Thus:

$$U_0(x) = \begin{cases} x - \frac{\cosh(\varphi - k)}{\varphi \cosh k} \sinh \varphi x \quad , & 0 \leq x \leq 1 \\ 1 - \frac{\sinh \varphi}{\varphi \cosh k} \cosh(\varphi x - k) \quad , & 1 \leq x \leq \lambda \end{cases}$$

Since $C_1 = v_0'$ we obtain

$$C_1(x) = \begin{cases} 1 - \dfrac{\cosh(\varphi - k)}{\cosh k} \cosh \varphi x \\ -\dfrac{\sinh \varphi}{\cosh k} \sinh(\varphi x - k) \end{cases}$$

Thus
$$C(x) \approx \begin{cases} 1 + \varepsilon\left(1 - \dfrac{\cosh(\varphi - k)}{\cosh k} \cosh \varphi x\right), & 0 \le x \le 1 \\ 1 + \varepsilon\left(\dfrac{\sinh \varphi}{\cosh k} \sinh(k - \varphi x)\right), & 1 \le x \le \lambda \end{cases}$$

The velocity at the outlet of the channel $(x = \lambda)$ is

$$v(\lambda) \approx v_0(\lambda) = 1 - \frac{\sinh \varphi}{\varphi \cosh k}.$$

Thus, the physical rate of volume of water flowing out of the channel is

$$v^*(L)A \approx V v_0(\lambda) A$$

$$= \frac{N_0 c \delta}{C_0}\left(1 - \frac{\sinh \varphi}{\varphi \cosh k}\right).$$

Here, $\varphi = \dfrac{k}{\lambda}$, $\lambda = L/\delta$, $k = \dfrac{\lambda}{\sqrt{\varepsilon \gamma}} = \left(\dfrac{cPC_0L^2}{AD}\right)^{1/2}$.

I think I stop here. Read more in Krogstad and Lin & Segel.