Mathematical modelling   Week 38, 2019

Singular perturbation (Lin & Segel, chap. 9)

As in regular perturbation, singular perturbation also involves a small parameter, say ε. The telltale property of a problem that requires singular perturbation rather than regular perturbation is that the problem changes in a qualitative way when we set $\varepsilon = 0$. E.g. the second degree equation

$$\varepsilon m^2 + 2m + 1 = 0 \quad , \quad 0 < \varepsilon << 1 ,$$

has two solutions $m_1(\varepsilon), m_2(\varepsilon)$. However, $\varepsilon = 0$ gives a first degree equation with only one solution $m_1 = -1/2$. And we can find estimates for $m_1(\varepsilon)$, for $0 < \varepsilon << 1$, using regular perturbation since εm^2 is small for $m \approx -1/2$.

Example $\quad \varepsilon m^2 + 2m + 1 = 0$

How about the other root $m_2(\varepsilon)$ for $0 < \varepsilon << 1$?

We understand that $m_2(\varepsilon)$ can not stay bounded as $\varepsilon \to 0$, since $|m_2(\varepsilon)| \leq M$, and εm^2 would be for $m \approx m_2(\varepsilon)$, but then $2m_2(\varepsilon)+1 \approx 0 \Leftrightarrow m_2(\varepsilon) \approx -1/2$

Thus $\varepsilon m_2(\varepsilon)^2$ is not small when $\varepsilon \to 0$. So we have to keep εm^2 in the equation. We want to find which term is "small" when εm^2 is not small.

1. Assume $2m$ is small compared to 1. Then

$$\varepsilon m^2 + 1 \approx 0 \Leftrightarrow m \approx \pm i \frac{1}{\sqrt{\varepsilon}} .$$

But then $2m \approx \pm i \frac{2}{\sqrt{\varepsilon}}$ is not small as $\varepsilon \to 0$

2. Assume 1 is small, so $\varepsilon m^2 + 2m \approx 0$. $m=0$ is no root (εm^2 not small) $\Rightarrow$ $\underline{m \approx -\frac{2}{\varepsilon}}$.

   We see that $\varepsilon m^2 \approx \frac{4}{\varepsilon}$, $2m \approx -\frac{4}{\varepsilon}$, $1=1$ meaning that the terms εm^2 and $2m$ balances, and that 1 is the small term.

Instead of setting $m = -\frac{2}{\varepsilon} + m_0 + m_1\varepsilon + \dots$, we will use an iteration scheme to obtain rational approximations to $m_2(\varepsilon)$.

<u>Iteration scheme</u>

(Note that this scheme also works for differential equations) We have an equation $F(m,\varepsilon)=0$, $0<\varepsilon \ll 1$.

Write $F(m,\varepsilon)=0 \Leftrightarrow f(m,\varepsilon) = g(m,\varepsilon)$, where

$f(m^0; \varepsilon) = 0$. m^0 is the approximation we have when setting the small term to zero in $F(m, \varepsilon) = 0$ (in the example $m^0 = -\frac{2}{\varepsilon}$). Then define $m^{(1)}, m^{(2)}, \ldots$ by

$$f(m^{(i+1)}) = g(m^{(i)}), \quad i = 0, 1, 2, \ldots$$

Then $\lim_{i \to \infty} m^{(i)} = m$ (if it exists) satisfies

$$f(m) = g(m).$$

Example, cont.

So, $\varepsilon m^2 + 2m + 1 \Leftrightarrow \varepsilon m^2 + 2m = -1$

$$\Leftrightarrow \varepsilon m + 2 = -\frac{1}{m} \quad \text{(make } f \text{ as simple as possible)}$$

$$\Rightarrow \varepsilon m^{(i+1)} + 2 = -\frac{1}{m^{(i)}} \Leftrightarrow m^{(i+1)} = \frac{1}{\varepsilon}\left(-2 - \frac{1}{m^{(i)}}\right)$$

$$\Leftrightarrow m^{(i+1)} = -\frac{2}{\varepsilon} - \frac{1}{\varepsilon m^{(i)}}.$$

This gives $m^{(1)} = -\frac{2}{\varepsilon} - \frac{1}{\varepsilon(-2/\varepsilon)} = -\frac{2}{\varepsilon} + \frac{1}{2}$

$$m^{(2)} = -\frac{2}{\varepsilon} - \frac{1}{\varepsilon\left(-\frac{2}{\varepsilon} + \frac{1}{2}\right)} = -\frac{2}{\varepsilon} + \frac{1}{2}\left(\frac{1}{1 - \varepsilon/4}\right).$$

$$= -\frac{2}{\varepsilon} + \frac{1}{2}\left(1 + \varepsilon/4 + O(\varepsilon^2)\right)$$

$$= -\frac{2}{\varepsilon} + \frac{1}{2} + \frac{\varepsilon}{8} + O(\varepsilon^2)$$

Singular perturbation, differential equations

Recall the example with the sinking object when friction was high. The scaled version of the equation was

$$\varepsilon x'' = -x' + 1 \quad , x(0) = 0 \ , x'(0) = \mu \ , 0 < \varepsilon \ll 1 .$$

Setting $\varepsilon = 0$ gives $-x' + 1 = 0$, a first order equation.

Thus, if not $\mu = 1$, the problem $-x' + 1 = 0$, $x(0) = 0$, $x'(0) = \mu$ has no solution. Physically there are two time regimes; the time from start to the time the ball reaches it's equilibrium velocity $x' = 1$ (rember friction is high), and the time intervall after when gravity and friction forces balances.

Such division of of the independent variable (here t) into two scales, or regimes, is typical for problems that need singular perturbation rather than regular perturbation. We demonstrate this for a concrete example.

Model example, discussion of the exact solution

Let $1 \gg \varepsilon > 0$, and consider: $\varepsilon y'' + 2y' + y = 0 \quad , y(0) = 0, \ y(1) = 1 .$

Characteristic equation is $\varepsilon m^2 + 2m + 1 = 0$, with (real) solutions $m_1(\varepsilon)$ and $m_2(\varepsilon)$. Thus,

$$y(x) = \frac{e^{m_1 x} - e^{m_2 x}}{e^{m_1} - e^{m_2}}.$$

From the last example we have $m_1 \approx -\frac{1}{2}$, $m_2 \approx -\frac{2}{\varepsilon}$, thus

$$y(x) \approx \frac{e^{-\frac{1}{2}x} - e^{-2x/\varepsilon}}{e^{-1/2} - e^{-2/\varepsilon}},$$

Since $e^{-2/\varepsilon} \approx 0$, we may write

$$y(x) \approx e^{1/2}\left(e^{-x/2} - e^{-2x/\varepsilon}\right).$$

We see $y(0) = 0$ and $y(1) \approx 1$. We also see that the two terms behave quite differently, $e^{-x/2} \approx 1$ for $x \in [0,1]$, while all significant change for $e^{-2x/\varepsilon}$ occurs in an interval $[0,\delta]$, where $\delta \approx \varepsilon$. For $x > \delta$ we have $e^{-2x/\varepsilon} \approx 0$. The interval $[0,\delta]$ is often referred to as a <u>boundary layer</u>. When ε is small, and hence δ small, the term $e^{1/2}e^{-x/2}$ approximate $y(x)$ well over the whole intervall <u>except</u> at the thin boundary layer. In singular perturbation the term $e^{1/2}e^{-x/2} \approx y_0(x)$ is the <u>outer solution</u>, while the term $-e^{1/2}e^{-2x/\varepsilon} = y_I(x)$ is the <u>inner solution</u>

Singular pertubation approach

Let's pretend we do not know how to solve

$$\varepsilon y'' + 2y' + y = 0, \; y(0) = 0, \quad 0 < \varepsilon << 1.$$

We saw that by putting $\varepsilon = 0$ we obtained a solution approximating the full solution $y(x, \varepsilon)$ well, <u>except</u> when x is close to $x = 0$, in the boundary layer.

For this $\varepsilon = 0$ solution we know that $\varepsilon y''(x, \varepsilon)$ is small when ε is small, justifying that $2y' + y = -\varepsilon y'' \approx 0$

<u>Outer solution</u>:

Setting $\varepsilon = 0$ we obtain $2y' + y = 0 \Rightarrow y_0 = Ce^{-x/2}$

Since not both $y(0) = 0$, $y(1) = 1$ can not be satisfied, we must choose one condition. Since y_0 is approximating $y(x, \varepsilon)$ well for $x > \delta$ we use $y(1) = 1$ as condition giving

$$\underline{y_0(x) \approx e^{1/2} e^{-x/2}}$$

By putting $y_0(x) = y^{(0)}(x) + y^{(1)}(x)\varepsilon + y^{(2)}(x)\varepsilon^2 + \ldots$ and assuming $\varepsilon y_0''$ is small, we could also get higher order approximations for $y_0(x)$.

Inner solution

Now we know that the term $\varepsilon y''(x,\varepsilon)$ is important when $\varepsilon \to 0$, and it's magnitude should should balance one of the two other terms, and the omitted term must be small compared to $\varepsilon y''$. If $2y'$ is not important, we have $\varepsilon y'' + y \approx 0 \Rightarrow y = e^{\pm i x/\varepsilon^{1/2}}$, so $|y|$ is bounded, and $y' = \pm \frac{i}{\varepsilon^{1/2}} e^{\pm i x/\varepsilon^{1/2}}$ unbounded as $\varepsilon \to 0$. Thus $2y'$ dominate y, contradiction.

Thus $\varepsilon y'' + 2y' = 0$ is the equation approximating $y(x,\varepsilon)$ when $\varepsilon y''$ is not small.

Scaling argument instead

Let $[0, \delta(\varepsilon)]$ be the boundary layer, and define
$Y(\xi, \varepsilon) = y(\delta(\varepsilon)\xi, \varepsilon)$ (i.e. when ξ goes from 0 to 1, x goes from 0 to $\delta(\varepsilon)$). Since Y typically is from 0 to 1 when $\xi \in [0,1]$, we can assume Y'', Y' and Y to be of order 1.

We have $y(x,\varepsilon) = Y(\frac{x}{\delta(\varepsilon)}, \varepsilon)$, so that Y satisfies

$$\frac{\varepsilon}{\delta^2} Y'' + \frac{2}{\delta} Y' + Y = 0 . \quad (*)$$

Thus $\frac{\varepsilon}{\delta^2}$, $\frac{2}{\delta}$, and 1 signify the magnitude of each term. So if $\frac{\varepsilon}{\delta^2} \approx 1 \Rightarrow \delta = \varepsilon^{1/2}$. Thus $\frac{2}{\delta} \to \infty$ as $\varepsilon \to 0$.

Thus $2y'$ would be large if $\varepsilon y''$ and y are of order 1.

Thus, $\frac{\varepsilon}{\delta^2} \approx 1$ is wrong. So, we have $\frac{\varepsilon}{\delta^2} \approx \frac{2}{\delta}$, and the boundary layer $\delta = \varepsilon$ (factor 2 not important)

From (*) we get (with $\delta = \varepsilon$) $Y'' + 2Y' + \varepsilon Y = 0$ as equation for Y. Again we could use regular perturbation to obtain higher order estimates for Y, but we settle for the first approximation, setting $\varepsilon = 0$ we get

$$Y = C_1 + C_2 e^{-2\xi}, \text{ and}$$

$$y(0) = 0 \Rightarrow Y(0) = 0 \Rightarrow Y(\xi) = C_1(1 - e^{-2\xi}).$$

Here we use the condition $y(0) = 0$ since the inner solution approximates $y(x, \varepsilon)$ close to $x = 0$.

$Y(\xi) = y_I(\xi)$ is the <u>inner solution</u>.

<u>Matching, uniform solution</u>

We determine C_1 by requiring that

$$y_I\left(\frac{x}{\delta(\varepsilon)}\right) = y_0(x)$$

for $x = \Theta(\varepsilon)$ in the limit $\varepsilon \to 0$. $\Theta(\varepsilon)$ should be "just outside" the boundary layer where $y_I\left(\frac{x}{\delta(\varepsilon)}\right)$ and $y_0(x)$ both are "good" approximations to $y(x,\varepsilon)$.

Actually we need not specify $\Theta(\varepsilon)$, we only require it to satisfy

$$\lim_{\varepsilon \to 0^+} \Theta(\varepsilon) = 0 \qquad (1)$$

$$\lim_{\varepsilon \to 0^+} \frac{\delta(\varepsilon)}{\Theta(\varepsilon)} = 0 \qquad (2)$$

Setting $y_0(\Theta(\xi)) = y_I\left(\frac{\Theta(\varepsilon)}{\delta(\sigma)}\right)$, or rather

$$\lim_{\varepsilon \to 0} y_0(\Theta(\xi)) = \lim_{\varepsilon \to 0^+} y_I\left(\frac{\Theta(\varepsilon)}{\delta(\xi)}\right)$$

and using (1) & (2):

$$\underline{\lim_{x \to 0} y_0(x) = \lim_{\xi \to \infty} y_I(\xi) = \bar{y}}$$

we have the matching condition. For us we get

$$\lim_{x \to 0} e^{1/2} e^{-x/2} = \lim_{\xi \to \infty} C_1 (1 - e^{-2\xi})$$

$\Leftrightarrow$ $\underline{C_1 = e^{1/2}}$, giving $\underline{y_I(\xi) = e^{1/2}(1 - e^{-2\xi})}$

The <u>uniform solution</u> approximates $y(x,\varepsilon)$ over the whole $[0,1]$, and is given by

$$\underline{y_u(x) = y_0(x) + y_I\left(\frac{x}{\delta}\right) - \bar{y}}$$

Thus our uniform solution is

$$y_u(x) = e^{1/2} e^{-x/2} + e^{1/2}\left(1 - e^{-2\frac{x}{\varepsilon}}\right) - e^{1/2}$$

$$= e^{1/2}\left(e^{-x/2} - e^{-2x/\varepsilon}\right),$$

the same approximate solution as previously obtained.

<u>Chapter 10, Lin & Segel</u>

<u>Singular perturbation and biochemical kinetics</u>

We will consider the dynamics, or kinetics, of a chemical reaction of a substance S to a substance P where an enzyme E is involved in the reaction. The mechanism of how E facilitates this reaction has been studied for a long time, and is thought to be illustrated by the following diagram:

$$S + E \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} C \xrightarrow{k_2} P + E$$

The diagram says that one S and one E combines to make a "complex" C at a reaction rate k_1, and one C splits to one P and E at rate k_2, or splits into one S and one E at rate k_{-1}.

Let $s^\infty(t^\infty)$, $e^\infty(t^\infty)$, $c^\infty(t^\infty)$, and $p^\infty(t^\infty)$ be respective molar concentrations at time t^∞ of the substances.

When two substances (or molecules) A and B combine to C, the reaction rate will increas proportionally to the concentration of both A and B (higher concentration of either increases the number of encounters of A and B which is necessary for A and B to react to C.

Thus $\dfrac{dc^\infty}{dt^\infty} = k a^\infty b^\infty$

where $a^\infty, b^\infty, c^\infty$ are concentrations of A, B, C.
k is the rate constant (note that for $A+B \xrightarrow{k} C$,
$[k] = \dfrac{1}{s \cdot mol}$ in SI-units.

From our diagram we then deduce the following:

$$\frac{ds^\infty}{dt^\infty} = -k_1 e^\infty s^\infty + k_{-1} c^\infty \qquad (1)$$

$$\frac{de^\infty}{dt^\infty} = -k_1 e^\infty s^\infty + k_{-1} c^\infty + k_2 c^\infty \qquad (2)$$

$$\frac{dc^\infty}{dt^\infty} = k_1 e^\infty s^\infty - k_{-1} c^\infty - k_2 c^\infty \qquad (3)$$

$$\frac{dp^\infty}{dt^\infty} = k_2 c^\infty \qquad (4)$$

$s^*(0) = s_0$, $e^\infty(0) = e_0$, $c^\infty(0) = p^\infty(0) = 0$

The last two initial conditions are zero, since at $t^\infty = 0$

we have not made any C and P yet. Usually $e_0 << S_0$ (much less E than S, it is S that is converted to P, while e^* is small and approximately constant).

Below we will use

$$\varepsilon = \frac{e_0}{S_0}$$

as our small dimensionless number in the perturbation analysis. We want to reduce the number of equations: If we add (2) and (3) we obtain

$$\frac{de^*}{dt^*} + \frac{dc^*}{dt^*} = 0 \iff e^*(t^*) + c^*(t^*) = \text{const}.$$

i.e. $e^*(t^*) + c^*(t^*) = e^*(0) + c^*(0) = e_0$

Thus $\underline{e^*(t) = e_0 - c^*(t)}$ (5)

Also, add (1), (3), and (4) we obtain

$$s^*(t^*) + c^*(t^*) + p^*(t) = \text{const} = S_0, \text{ so}$$

$$\underline{p^*(t^*) = S_0 - S^*(t^*) - c^*(t^*)}$$

Using (5) in (1) and (3) we obtain

$$\frac{dS^*}{dt^*} = -k_1(e_0 - c^*)S^* + k_{-1}c^*$$

$$\frac{dc^*}{dt^*} = k_1(e_0 - c^*)S^* - k_{-1}c^* - k_2c^*,$$

rearranging, we obtain

$$\frac{ds^*}{dt^*} = -k_1 e_0 s^* + (k_1 s^* + k_{-1}) c^* \qquad (6)$$

$$\frac{dc^*}{dt^*} = k_1 e_0 s^* - (k_1 s^* + k_{-1} + k_2) c^* \qquad (7)$$

$$s^*(0) = s_0 \, , \quad c^*(0) = 0$$

as our initial value problem.

Note that $[k_{-1}] = [k_2] = \frac{1}{s}$, while $[k_1] = \frac{1}{mol \cdot s}$.

At the start of the reaction c^* will rise (rapidly usually) to a value of the same order as e_0.

Scaling

As just mentioned, it is reasonable to put

$$c^* = e_0 c \, .$$

Furthermore, $s^* = s_0 s$ is natural. Let us put

$$t^* = Tt \, ,$$

and find T by balancing the equations.

From (6) we get

$$\frac{s_0}{T} s' = -k_1 e_0 s_0 s + (k_1 s_0 s + k_{-1}) e_0 c$$

$$\Leftrightarrow \quad s' = -(Tk_1 e_0) s + ((e_0 k_1 T) s + \frac{T e_0 k_{-1}}{s_0}) c$$

Put $T = \frac{1}{e_0 k_1}$ giving

$$s' = -s + \left(s + \frac{k_{-1}}{k_1 s_0}\right)c$$

For notational convenience let $\frac{k_{-1}}{k_1 s_0} = k - \lambda$, where

$$\underline{k = \frac{k_{-1} + k_2}{k_1 s_0}} \quad \text{and} \quad \underline{\lambda = \frac{k_2}{k_1 s_0}}$$

Thus, we have $\underline{s' = -s + (s + k - \lambda)c}$.

$$\frac{e_0}{\left(\frac{1}{e_0 k_1}\right)} c' = k_1 e_0 s_0 s - (k_1 s_0 s + k_{-1} + k_2) e_0 c$$

$$\Leftrightarrow \quad e_0 k_1 c' = k_1 s_0 s - (k_1 s_0 s + k_{-1} + k_2) c$$

$$\Leftrightarrow \quad \frac{e_0}{s_0} c' = s - \left(s + \frac{k_{-1} + k_2}{s_0 k_1}\right) c$$

$$\Leftrightarrow \quad \underline{\varepsilon c' = s - (s + k) c},$$

where $\underline{\varepsilon = e_0/s_0}$. To sum up:

$$\left.\begin{aligned} s' &= -s + (s + k - \lambda)c \\ \varepsilon c' &= s - (s + k)c \\ c(0) &= 0,\ s(0) = 1 \end{aligned}\right\}$$

where $0 < \varepsilon << 1$.

Hence (due to $\varepsilon c'$) this set of equations require singular perturbation for obtaining approximate solution.

Lets find an approximation for the <u>outer solution</u> by setting

$\varepsilon = 0$: Thus

$$0 = s - (s+k)c \quad \text{, giving}$$

$$c = \frac{s}{s+k} \qquad (8)$$

Insert this into the first equation to obtain

$$s' = -s + (s+k-\lambda)\frac{s}{s+k} \iff$$

$$s' = -\frac{\lambda s}{s+k} \quad \text{, which is separable :}$$

$$\Rightarrow \left(1 + \frac{k}{s}\right) ds = -\lambda\, dt \iff$$

$$\underline{s + k \ln s = -\lambda t + K_1} \;,\; K_1 \text{ constant.} \qquad (9)$$

Thus (8) and (9) define (implicitly) the zero order ($\varepsilon = 0$) approximation to the <u>outer solution</u> $s_0(t)$ and $c_0(t)$. The constant K will be found by the matching condition.

<u>So, lets consider the inner solution</u>

Since $t \in [0, \infty)$ we must have the boundary layer close to $t = 0$. We now know that $\varepsilon c'$ is <u>not</u> small compared to the other terms. As usual, put

$$s(t;\varepsilon) = \hat{s}\left(\frac{t}{\delta(\varepsilon)}, \varepsilon\right) \text{ and } c(t;\varepsilon) = \hat{c}\left(\frac{t}{\delta(\varepsilon)}, \varepsilon\right),$$

where $\xi = \frac{t}{\delta(\varepsilon)}$. Inserted into the equations:

$$\left.\begin{aligned} \frac{\hat{s}'}{\delta} &= -\hat{s} + (\hat{s}+k-\lambda)\hat{c} \\ \frac{\varepsilon}{\delta}\hat{c}' &= \hat{s} - (\hat{s}+k)\hat{c} \end{aligned}\right\} \quad \hat{s}(0,\varepsilon)=1, \; \hat{c}(0,\varepsilon)=0$$

Since $\hat{s}$ is of order 1, we have $\frac{\varepsilon}{\delta}$ order 1. Thus $\varepsilon=\delta$ for the boundary layer. This gives

$$\hat{s}' = \varepsilon\left[-\hat{s} + (\hat{s}+k-\lambda)\hat{c}\right]$$

$$\hat{c}' = \hat{s} - (\hat{s}+k)\hat{c}$$

Set $\varepsilon=0$ to find the zero order approximation to the inner solutions:

$$\hat{s}'=0 \Rightarrow \hat{s}(\xi)=K_2, \text{ constant.}$$

Since we are in the boundary layer, we use $\hat{s}(0)=1$, $\hat{c}(0)=0$, so $\hat{s}\equiv 1$. Insert this in the next equation:

$$\hat{c}' = 1-(1+k)\hat{c}, \text{ giving}$$

$$\hat{c} = \frac{1}{1+k} + K_2 e^{-(1+k)\xi}$$

$\hat{c}(0)=0$ gives $\quad \hat{c}(\xi) = \frac{1}{1+k}\left(1-e^{-(1+k)\xi}\right).$

So, summing up: To zero order we have

Outer solution: $\begin{cases} c_0(t) = \dfrac{s_0(t)}{s_0(t)+k} \\ s_0(t) + k\ln s_0(t) = -\lambda t + K_1 \end{cases}$ Inner sol: $\begin{cases} c_I(\xi) = \dfrac{1-e^{-(1+k)\xi}}{1+k} \\ s_I(\xi) = 1 \end{cases}$

Matching:

We must have $\lim_{t\to 0} S_0(t) = \lim_{\xi\to\infty} S_I(\xi) = 1 = \bar{S}$

so $S_0(0) = 1$. From $S_0 + k\ln S_0 = -\lambda t + K$, we get

$$1 + k\ln 1 = -\lambda\cdot 0 + K_1 \Rightarrow \underline{K_1 = 1} \qquad (10)$$

Then $\lim_{t\to 0} C_0(t) = \lim_{\xi\to\infty} C_I(\xi) = \bar{C}$

$$\Leftrightarrow \quad \frac{1}{1+k} = \frac{1}{1+k} \quad \text{automatically satisfied!}$$

So, the uniform solution for S is

$$S_u(t) = S_0(t) + S_I\left(\frac{t}{\delta}\right) - 1$$

$$\Leftrightarrow \quad \underline{S_u(t) = S_0(t)} \quad \text{(given implicitly by (10))}$$

For c: $C_u(t) = C_0(t) + C_I\left(\frac{t}{\delta}\right) - \bar{C} \quad \Leftrightarrow$

$$C_u(t) = \frac{S_0(t)}{S_0(t)+k} + \frac{1}{1+k}\left(1 - e^{-(1+k)\frac{t}{\epsilon}}\right) - \frac{1}{1+k}$$

$$\Leftrightarrow \quad \underline{C_u(t) = \frac{S_0(t)}{S_0(t)+k} - \frac{1}{1+k} e^{-(1+k)t/\epsilon}}$$

Estimates for $S_a(t)$:

We have $S_0(t) + k\ln S_0(t) = -\lambda t + 1$, where $S_0(0) = 1$

$$\Rightarrow S_0'(t) + \frac{k S_0'(t)}{S_0(t)} = -\lambda \text{ . So for } t=0 \ (S_0(0)=1)$$

we have $S_0'(0) = -\frac{\lambda}{1+k}$. Thus

$$S_0(t) \simeq 1 - \frac{\lambda}{1+k} t = S_u(t) \text{ , giving}$$

$$C_0(t) = \frac{S_0(t)}{S_0(t)+k} \simeq \frac{1-\frac{\lambda}{1+k}t}{1-\frac{\lambda}{1+k}t+k} = \frac{1+k-\lambda t}{(1+k)^2-\lambda t}$$

Thus $$C_u(t) \simeq \frac{1+k-\lambda t}{(1+k)^2-\lambda t} - \frac{1}{1+k} e^{-(1+k)t/\varepsilon}$$

These expressions are valid when t is "small".