Mathematical modelling, week 39, 2019

(Litterature: D. J. Logan, Applied Mathematics, chapter 6)

Stability

Consider a system of first order equations of the form

$$(*)\quad \left.\begin{aligned} \frac{dx_1}{dt} &= f_1(x_1, x_2, \dots, x_n) \\ \frac{dx_2}{dt} &= f_2(x_1, \dots, x_n) \\ &\vdots \\ \frac{dx_n}{dt} &= f_n(x_1, \dots, x_n) \end{aligned}\right\} \Leftrightarrow \frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x}(t) \in \mathbb{R}^n. \quad \mathbf{x}(0) = \mathbf{x}_0$$

where $f_1, \dots, f_n$ are given functions of n variables. Let $x_j(0) = x_j^0$, $j = 1, \dots, n$, be initial values. We assume $(*)$ has a unique solution for $t > 0$ for any initial values $x_j^0(0)$, $j = 1 \dots n$. $(*)$ is often referred to as a <u>dynamical system</u>. Note that geometrically $\mathbf{f}(\mathbf{x})$ is the tangent vector to the solution curve $\mathbf{x}(t) \in \mathbb{R}^n$.

<u>Definition</u> An <u>equilibrium point</u> $\mathbf{x}_e \in \mathbb{R}^n$ is a constant solution to $(*)$, i.e. $\mathbf{x}(t) = \mathbf{x}_e$ is a solution.

Thus $\mathbf{x}_e$ is an equilibrium point if and only if $\mathbf{f}(\mathbf{x}_e) = \mathbf{0}$.

Typically, if a dynamic system settles in a point $\mathbf{x}_e$

as $t \to \infty$, then x_e is an equilibrium point.

Equilibrium points can be categorized as follows:

* The equilibrium point x_e is <u>stable</u>, if for all $\varepsilon > 0$ there exist a $\delta > 0$, such that if
$$\| x_0 - x_e \| < \delta$$
then $\| x_0 - x(t) \| < \varepsilon$ for all $t > 0$.

* The equilibrium point x_e <u>asymptotically stable</u> if it is stable and if there is an $\varepsilon > 0$ such that
$$\| x_0 - x_e \| < \varepsilon \text{ and } x(t) \text{ solves } x' = f(x),\ x(0) = x_0$$
implies that $\| x(t) - x_0 \| \to 0$ as $t \to \infty$.

* The equilibrium point is <u>unstable</u> if it is not stable

<u>Example</u>

Consider the 1-dimensional example (the logistic equation)
$$(*) \qquad x' = ax - bx^2, \quad x(0) = x_0, \quad a > 0, b > 0.$$

Here $f(x) = ax - bx^2 = 0 \Leftrightarrow x = 0$ or $x = a/b$.

Thus we have two equilibrium points.

<u>Consider first $x_e = a/b$:</u>

Let $x(t)$ be the solution having $x(0) = x_0$. Define $\hat{x}(t)$ by
$$x(t) = x_e + \hat{x}(t),$$

and find the equation $\hat{x}(t)$ satisfies: From (*)

$$(x_e + \hat{x})' = a(x_e + \hat{x}) - b(x_e + \hat{x})^2$$

$$\Leftrightarrow \hat{x}' = a\hat{x} + ax_e - bx_e^2 - 2x_e b\hat{x} - b\hat{x}^2$$

$$\Leftrightarrow \hat{x}' = ax_e - bx_e^2 + (a - 2x_e b)\hat{x} - b\hat{x}^2$$

$x_e = \frac{a}{b}$ gives $\underline{\hat{x}' = -a\hat{x} - b\hat{x}^2}$ (**)

If $|\hat{x}(0)| = |\hat{x}_0|$ is small (i.e. $|x_0 - x_e|$ small), then $\hat{x}^2$ is small, and ignoring $-b\hat{x}^2$ we obtain

$$\hat{x}' = -a\hat{x} \Rightarrow \hat{x}(t) = \hat{x}(0)e^{-at}.$$

Thus $|\hat{x}(t)| = |x(t) - x_e| \to 0$, indicating that $x_e = a/b$ is an asymptotically stable equilibrium point when $|x_0 - x_e|$ is "small".

Now, it is actually possible to solve (**) exactly (it is separable) giving

$$\hat{x}(t) = \frac{(bc/a)e^{-bt}}{1 - ce^{-bt}}, \text{ where } c \text{ is a constant.}$$

Then $\hat{x}(t) \to 0$ as $t \to \infty$, meaning that $x_e = a/b$ is asymptotically stable.

$\underline{x_e = 0}$:

The other equilibrium point $x_e = 0$. So, let $x(t) = \hat{x}(t) + 0$.

Thus $\hat{x}' = a\hat{x} - b\hat{x}^2$, $\hat{x}(0) = \hat{x}_0$.

If $\hat{x}_0$ is small $\hat{x} \doteq c e^{at}$, so $|\hat{x}(t)| \to \infty$ as $t \to \infty$

The full equation has solution

$$\hat{x}(t) = \frac{ac}{bc + (a-c)e^{-at}} \to \frac{a}{b} \quad \text{as } t \to \infty$$

Thus $x_e = 0$ is unstable (since $\hat{x}(t) \not\to 0$).

Linear systems

Consider the linear system

$$\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 \in \mathbb{R}^n,$$

where A is a constant $n \times n$-matrix. The equilibrium points satisfy $A\mathbf{x}_e = \mathbf{0}$.

If A is invertible, we one single equilibrium point $\mathbf{x}_e = \mathbf{0}$.

The solution to $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = 0$ is

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0,$$

where $e^U = \sum_0^\infty \frac{U^n}{n!}$. If A is diagonalizable, then the general solution to $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x}(t) = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \mathbf{v}_2 e^{\lambda_2 t} + \dots + c_n \mathbf{v}_n e^{\lambda_n t} \qquad (*)$$

where $c_1, \dots, c_n$ are determined by the initial conditions, $\mathbf{v}_1, \dots, \mathbf{v}_n$ eigenvectors, and $\lambda_1, \dots \lambda_n$ corresponding eigenvalues.

Note that eigenvectors and eigenvalues are complex in general. We observe that if

(1) $$\operatorname{Re}\lambda_j < 0 \quad \text{for } j = 1, \dots, n,$$

then $x(t) \to 0$ always, and $x_e = 0$ is consequently <u>asymptotically stable</u>. If

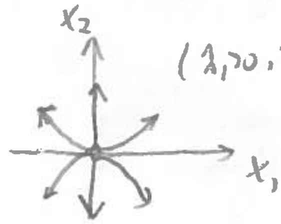
(2) $$\operatorname{Re}\lambda_k > 0 \quad \text{for at least one } k,$$

then x_e is an <u>unstable</u> equilibrium point.

(3) If $\operatorname{Re}\lambda_j \leq 0$, for $j = 1, \dots, n$, and $\operatorname{Re}\lambda_k = 0$ for one k (at least), then x_e is <u>stable</u>.

<u>Example $\mathbb{R}^2$:</u> If A is 2×2-matrix we have 4 typical behaviors:

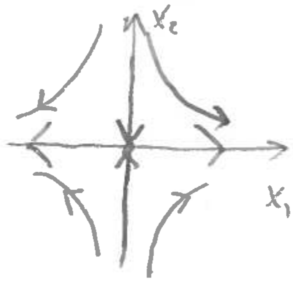
1) <u>Nodes</u> $\lambda_1 < 0, \lambda_2 < 0$ (stable); $\lambda_1 > 0, \lambda_2 > 0$ (unstable)



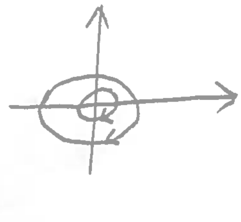
$(\lambda_1 > 0, \lambda_2 > 0)$

2) <u>Saddle point</u> $\lambda_1 < 0 < \lambda_2$

(unstable)



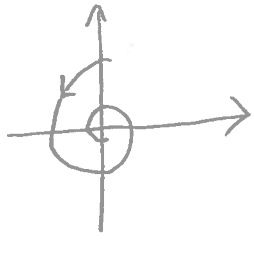
3) <u>Center</u> $\lambda_1, \lambda_2 = \pm i\omega$



(stable)

4) Spirals $\lambda_1, \lambda_2 = \alpha \pm i\omega$

$\alpha > 0$ (unstable), $\alpha < 0$ (stable)



$(\alpha < 0)$

Non-linear systems

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}_0) + d\mathbf{f}(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0) + O(\|\mathbf{x}-\mathbf{x}_0\|^2)$$

↑ Taylor series

where

$$d\mathbf{f} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$ is the Jacobi matrix

Let $\mathbf{x}_e$ be an equilibrium point, and put

$$\mathbf{x}(t) = \hat{\mathbf{x}}(t) + \mathbf{x}_e$$

Then $$\hat{\mathbf{x}}' = \mathbf{f}(\hat{\mathbf{x}} + \mathbf{x}_e) = \underbrace{\mathbf{f}(\mathbf{x}_e)}_{=0} + D\mathbf{f}(\mathbf{x}_e)\hat{\mathbf{x}} + O(\|\hat{\mathbf{x}}\|^2)$$

The linearized system is

$$\hat{\mathbf{x}}' = A\hat{\mathbf{x}} \quad , \text{where } A = d\mathbf{f}(\mathbf{x}_e).$$

__Theorem:__ If f is twice continuously differentiable, and λ_i eigenvalues of $A = d\mathbf{f}(\mathbf{x}_e)$, then

a) $\max \operatorname{Re} \lambda_i < 0 \Rightarrow \mathbf{x}_e$ asymptotically stable

b) $\max \operatorname{Re} \lambda_i = 0 \Rightarrow$ no conclusion

c) $\max \operatorname{Re} \lambda_i > 0 \Rightarrow$ unstable.

See problem set 5 and 6!

Example

Chemotaxis of amoebae

Let $a(x,t)$ be amoebae concentration, $[a] = m^{-3}$, and $\varphi(x,t)$ the flux of amoebae, $[\varphi] = m^{-2}s^{-1}$ (number of amoebae passing per area per time.

$$\varphi = \varphi_a + \varphi_c$$

where $\varphi_a = -k \frac{\partial a}{\partial x}$, $k > 0$ is a diffusion mechanism making amoebae spread from high concentration areas to lower concentration areas.

φ_c is the flux due to chemotaxis making amoebae migrate towards high concentration of a chemical with concentration $c(x,t)$. We set

$$\varphi_c = \ell a(x,t) \frac{\partial c}{\partial x}(x,t), \quad \ell > 0$$

Thus, $\varphi = -k \frac{\partial c}{\partial x} + \ell a \frac{\partial c}{\partial x}$.

Conservation of amoebae number:

We have $\varphi(x)A - \varphi(x+\Delta x)A$ = rate of accumulation of amoebae in interval $[x, x+\Delta x]$

Thus $\frac{d}{dt}(a \Delta x A) = \varphi(x)A - \varphi(x+\Delta x)A$, and let $\Delta x \to 0$, giving

$$\frac{\partial a}{\partial t} + \frac{\partial \varphi}{\partial x} = 0$$

Thus, $\frac{\partial a}{\partial t} = \frac{\partial}{\partial x}\left(k \frac{\partial a}{\partial x} - l a \frac{\partial c}{\partial x}\right)$.

Then we need to state mass conservation for chemical:

The flux is $\varphi_d = -D \frac{\partial c}{\partial x}$.

Now we also have a source term $Q = a q_1 - c q_2$

where q_1 is the rate one amoeba generates the chemical, and q_2 is the rate of decay. In the interval $[x, x+\Delta x]$ we have

$$\frac{d}{dt} \int_x^{x+\Delta x} c(s,t) A \, ds = \varphi_d(x,t) A - \varphi_d(x+\Delta x, t) + \int_x^{x+\Delta x} Q(s,t) \, ds$$

$$\Rightarrow \frac{\partial c}{\partial t} + \frac{\partial}{\partial x} q_d = Q \quad \text{, giving}$$

$$\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2} + a q_1 - c q_2 .$$

Consequently,

$$\frac{\partial a}{\partial t} = \frac{\partial}{\partial x}\left(k \frac{\partial a}{\partial x} - l a \frac{\partial c}{\partial x}\right)$$

$$\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2} + a q_1 - c q_2$$

We see that $a(x,t) = a_0$, $c(x,t) = c_0$, provided that $a_0 q_1 - c_0 q_2 = 0$, gives equilibrium point.

Write $a(x,t) = a_0 + \tilde{a}(x,t)$

$c(x,t) = c_0 + \hat{c}(x,t)$.

This gives

$$\frac{d\hat{a}}{dt} = k\frac{d^2\hat{a}}{dx} - la_0\frac{d^2\hat{c}}{dx^2} - \frac{d}{dx}\left(\hat{a}\frac{d\hat{c}}{dx}\right)$$

$$\frac{d\hat{c}}{dt} = D\frac{d^2\hat{c}}{dx^2} + \hat{a}q_1 - \hat{c}q_2$$

Linearize, to obtain

$$\frac{d\hat{a}}{dt} = k\frac{d^2\hat{a}}{dx^2} - la_0\frac{d^2\hat{c}}{dx^2}$$

$$\frac{d\hat{c}}{dt} = D\frac{d^2\hat{c}}{dx^2} + \hat{a}q_1 - \hat{c}q_2$$

From Fourier-analysis, put $\hat{a} = c_1 e^{\alpha t} \cdot e^{i\beta x}$

$\hat{b} = c_2 e^{\alpha t} \cdot e^{i\beta x}$

giving $\alpha c_1 = -c_1 k\beta^2 + la_0 c_2 \beta^2$

$\alpha c_2 = -c_2 D\beta^2 + c_1 q_1 - c_2 q_2$

$\Leftrightarrow$ $(\alpha + k\beta^2)c_1 - la_0\beta^2 c_2 = 0$

$-q_1 c_1 + (\alpha + D\beta^2 + q_2)c_2 = 0$

This set of equations has non-trivial solution for c_1 and c_2 if and only if

$$\begin{vmatrix} \alpha + k\beta^2 & -la_0\beta^2 \\ -q_1 & \alpha + D\beta^2 + q_2 \end{vmatrix} = 0 \quad \Leftrightarrow$$

$$\alpha^2 + (k\beta^2 + D\beta^2 + q_2)\alpha + kD\beta^4 + k\beta^2 q_2 - la_0\beta^2 q_1 = 0$$

We get to real roots since the discriminant is

$$D = (k\beta^2 - D\beta^2 - q_2)^2 + 4q_1 l a_0 \beta^2 > 0$$

Do we have a positive root? For a 2-degree equation $x^2 + \beta x + \gamma = 0 \Leftrightarrow x = -\frac{\beta}{2} \pm \frac{1}{2}\sqrt{\beta^2 - 4\gamma}$. The largest root is

$$-\frac{\beta}{2} + \frac{1}{2}\sqrt{\beta^2 - 4\gamma} > 0 \Leftrightarrow \sqrt{\beta^2 - 4\gamma} > \beta.$$ So, for $\beta > 0$

(as we have), the largest root is positive if and only if $\gamma < 0 \Leftrightarrow kD\beta^4 + kq_2\beta^2 - la_0 q_1 \beta^2 < 0$

$\Leftrightarrow \underline{kD\beta^2 + kq_2 < la_0 q_1} \Rightarrow \alpha > 0$, hence instability.

The opposite inequality gives the criteria for stability.

* We see when the wave-number $\beta \to 0$ it is easier to get instability. For sufficiently large β we get stability.
* Also, when q_1 increases (due to lack of food), instability can be trigered, and we have onset of aggregation!

Bifurcation theory

When a dynamical system

$$x' = f(x, \mu), \quad x(0) = x_0$$

depends on a parameter $\mu \in \mathbb{R}$, the equilibrium points $x_e = x_e(\mu)$ depend on μ. And also stability properties of $x_e(\mu)$ can change as μ varies. In addition, the number of equilibrium points may depend on μ's value.

1-D theory

Consider

$$x' = f(x, \mu) \qquad (1)$$

where $\mu \in \mathbb{R}$ is a parameter. For fixed $\mu \in \mathbb{R}$, the equilibrium points $x_e = x_e(\mu)$ are the solutions to

$$f(x, \mu) = 0.$$

For certain values of μ_c the qualitative nature of (1) can change, either by (or both)

* $x_e(\mu)$ changes from stable to unstable, or vice versa.
* The number of equilibrium points changes.

μ_c is called a bifurcation point

Theorem

Let $x' = f(x,\mu)$, $\mu \in \mathbb{R}$, and let $x_e(\mu)$ be an equilibrium point. Assume

$$f(x_e + \tilde{x}, \mu) = \frac{\partial f}{\partial x}(x_e,\mu)\hat{x} + O(\hat{x}^2)$$

Then $x_e(\mu)$ is

a) Asymptotically stable if $\frac{\partial f}{\partial x}(x_e,\mu) < 0$

b) Unstable if $\frac{\partial f}{\partial x}(x_e,\mu) > 0$.

(Makes sense as the linearized problem is stable)

Example $x' = \mu x - x^2 = f(x,\mu)$

$f(x,\mu) = 0 \Leftrightarrow x = 0$ or $x = \mu$, and $\frac{\partial f}{\partial x} = \mu - 2x$

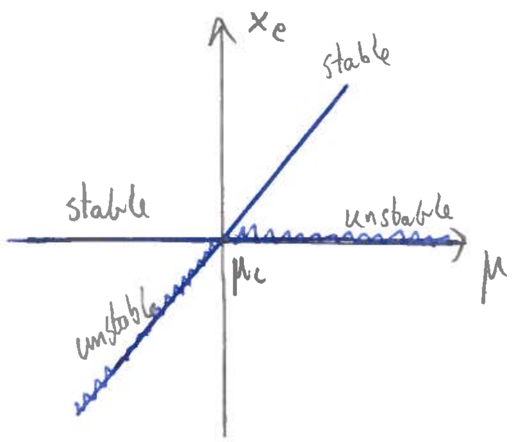
The equilibrium point $x_e = 0$ has $\frac{\partial f}{\partial x}(0,\mu) = \mu$.

Thus $x_e = 0$ is stable for $\mu < 0$ and unstable for $\mu > 0$.

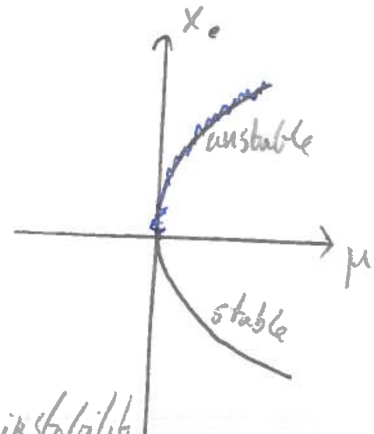
For $x_e = \mu$ we have $\frac{\partial f}{\partial x}(\mu,\mu) = \mu - 2\mu = -\mu$, so that $x_e = \mu$ is unstable for $\mu < 0$ and stable for $\mu > 0$.

We observe that the stability properties of $x' = f(x,\mu)$ changes when μ changes sign. $\mu = \mu_c = 0$ is consequently the only bifurcation point.

We may illustrate the properties of the equilibrium points in a bifurcation diagram:



Example 1 Regular turning point $\frac{df}{dx} = 0$, $\frac{df}{d\mu} \neq 0$



$$x' = x^2 - \mu = f(x, \mu).$$

Then $f(x_e, \mu) = 0 \iff x_e^2 = \mu$.

So for $\mu \geq 0$ we get $x_e = \pm\sqrt{\mu}$

Now $\frac{df}{dx} = 2x$, meaning that $+\sqrt{\mu}$ gives instability and $-\sqrt{\mu}$ is a stable equilibrium point.

Thus $\mu = 0$ is a bifurcation point.

In general, at a regular turning point (x_e, μ) we have $\frac{df}{dx} = 0$ and $\frac{df}{d\mu} \neq 0$, meaning that we can locally solve $f(x, \mu) = 0$ as $\mu(x) = \mu$, where

$$\frac{d\mu}{dx} = -\frac{df/dx}{df/d\mu} = 0.$$

The implicit function theorem tells us when $f(x,\mu)=0$ is locally a "normal" curve (i.e. no branching). We simply have:

(x,μ) a regular point

$$\begin{cases} \text{If } \frac{df}{dx}(x,\mu) \neq 0 & \text{we have } x = x(\mu) \text{ locally, and} \\ \text{if } \frac{df}{d\mu}(x,\mu) \neq 0 & \text{we can locally write } \mu = \mu(x). \end{cases}$$

Thus we need to have $\frac{df}{dx}(x,\mu) = \frac{df}{d\mu}(x,\mu) = 0$ in order to possibly having a branching point for the equation $f(x,\mu)=0$. In this case (x,μ) is called a <u>singular point</u>.

Double points

Let (x_0,μ_0) be a singular point. Then

$$f(x_0+\Delta x, \mu_0+\Delta\mu) = f(x_0,\mu_0) + f_x(x_0,y_0)\Delta x + f_\mu(x_0,\mu_0)\Delta\mu$$
$$+ \frac{1}{2}\left(f_{xx}\Delta x^2 + 2f_{x\mu}\Delta x\Delta\mu + f_{\mu\mu}\Delta\mu^2\right) + \text{higher order.}$$

If $f(x_0+\Delta x, \mu_0+\Delta\mu) = 0$ and (x_0,μ_0) is a singular point, then

$$f_{xx}\Delta x^2 + 2 f_{x\mu}\Delta\mu\Delta x + f_{\mu\mu}\Delta\mu^2 = 0$$

$$\Leftrightarrow f_{xx}\left(\frac{\Delta x}{\Delta\mu}\right)^2 + 2 f_{x\mu}\left(\frac{\Delta x}{\Delta\mu}\right) + f_{\mu\mu} = 0$$

Second degree equation for $\left(\frac{\Delta x}{\Delta\mu}\right)$, with discriminant

$$D = 4 f_{x\mu}^2 - 4 f_{xx} f_{\mu\mu}$$

If $D > 0$ and $f_{xx} \neq 0$ we have to real solutions for $\frac{\Delta x}{\Delta\mu}$:

$$\frac{\Delta x}{\Delta\mu} \approx \frac{dx}{d\mu} = \frac{1}{f_{xx}}\left(-f_{x\mu} \pm \left(f_{x\mu}^2 - f_{xx}f_{\mu\mu}\right)\right)$$

If $D > 0$ and $f_{\mu\mu} \neq 0$ we find two solutions for $\frac{d\mu}{dx}$. When $D > 0$ we have a double point

If $D < 0$ there are no real solutions, and the equilibrium $x_e(\mu)$ is an isolated point

If $D = 0$ there are at least 2 curves through the critical point having coincident tangents.

We saw that $(0,0)$ is a double point

for $\quad x' = \mu x - x^2 = f(x,\mu)$

We see that $f_x(0,0) = 0$, $f_\mu(0,0) = 0$

$f_{xx} = -2$, $f_{\mu x} = -1$, $f_{\mu\mu} = 0 \Rightarrow$

$D = 4(f_{x\mu}^2 - f_{xx} f_{\mu\mu}) = 4 > 0$, and

we obtain $\quad \dfrac{du}{d\mu} = 1$ or $\dfrac{du}{d\mu} = 0$

as seen in the first bifurcation diagram.

Example (Pitchfork)

$$x' = \mu x - x^3 = f(x,\mu)$$

$f(x,\mu) = 0 \Leftrightarrow x(\mu - x^2) = 0 \qquad x = 0$ or $x = \pm\sqrt{\mu}$

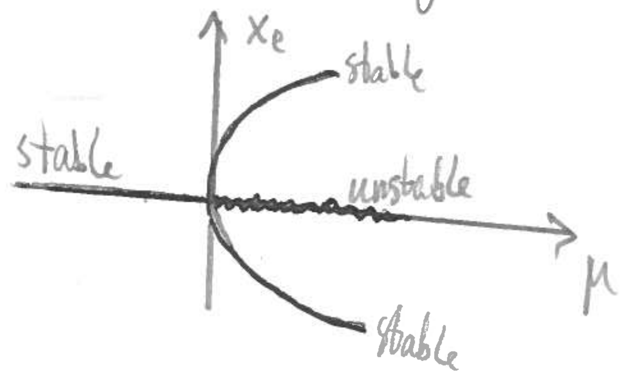
Thus, for $\mu < 0$, $x_e = 0$ only equilibrium point.

$\dfrac{df}{dx}(0,\mu) = \mu \Rightarrow (0,\mu)$ is stable for $\mu < 0$,

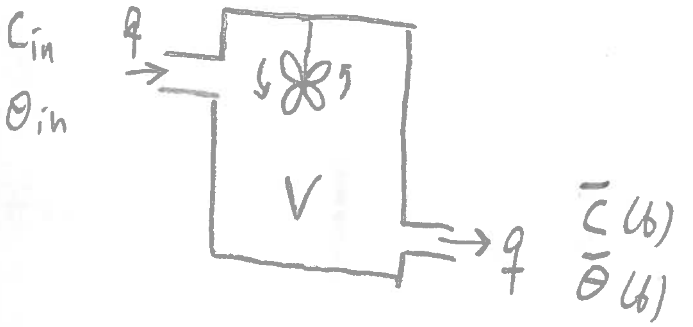
unstable for $\mu > 0$.

While $\dfrac{df}{dx}(\pm\sqrt{\mu}, \mu) = \mu - 3(\pm\sqrt{\mu})^2 = -2\mu < 0$ for $\mu > 0$

Bifurcation diagram



Tank reactor



V : tank volume

$\bar{c}$: concentration of chemical

$\bar{\theta}$: temperature

q = feed rate $[q] = m^3 s^{-1}$

c_{in} : concentration of input

θ_{in} : temperature of input

We feed a tank with a chemical having concentration c_{in} and temperature θ_{in} as it enters the tank.

We assume the chemical disappears at a rate $-k\bar{c}\, e^{-A/\bar{\theta}}$, and that the reaction gives of $hk\bar{c}\, e^{-A/\bar{\theta}}$ energy per volume and time. Here k, h, A are constants. The tank is stirred, so we assume uniform concentration and temperature in the tank

Mass conservation for chemical

$$\frac{d}{d\bar{t}}(V\bar{c}) = q c_{in} - q\bar{c} - Vk\bar{c}e^{-A/\bar{\theta}}$$

Heat balance, or conservation of thermal energy:

$$\frac{d}{d\bar{t}}(CV\bar{\theta}) = q c_{in} C - q\bar{\theta}C + hVk\bar{c}e^{-A/\bar{\theta}},$$

where C is heat capacity of the mixture

Scaling Put $\bar{t} = \frac{V}{q}t$, $\bar{\theta} = \theta_{in}\theta$, $\bar{c} = c_{in}c$,

giving

$$c' = 1 - c - \frac{c}{\mu}e^{-\gamma/\theta} \quad (1)$$

$$\theta' = 1 - \theta + \frac{bc}{\mu}e^{-\gamma/\theta} \quad (2)$$

$$\boxed{\mu = \frac{q}{kV} \quad b = \frac{hc_{in}}{C\theta_{in}} \quad \gamma = \frac{A}{\theta_{in}}}$$

Multiply (1) with b and add to (2) giving

$$(bc+\theta)' = 1+b-(bc+\theta)$$

$$\Rightarrow bc+\theta = 1+b+Ke^{-t}$$

Assuming $\theta(0)=1$, $c(0)=1$ we get $K=0$ giving

$$c = \frac{1}{b}(1+b-\theta).$$

Inserted into (2) gives

$$\theta' = 1 - \theta + \frac{b}{\mu}\left(\frac{1}{b}(1+b-\theta)\right)e^{-\gamma/\theta}$$

$$\Leftrightarrow \quad \theta' = 1-\theta + \frac{1-\theta+b}{\mu} e^{-\gamma/\theta}$$

Let $\underline{u = \theta - 1}$ giving

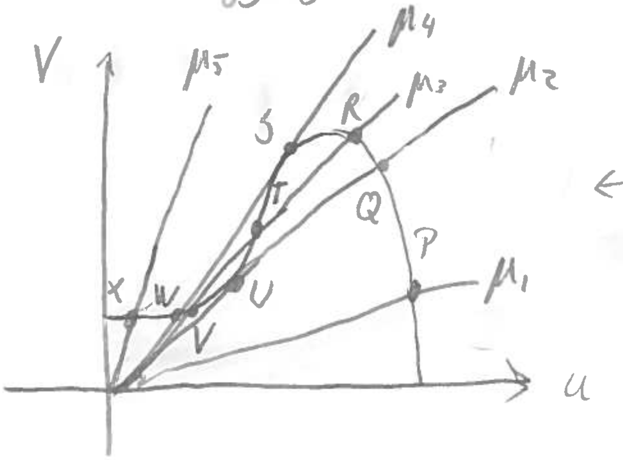
$$u' = -u + \frac{b-u}{\mu} e^{-\frac{\gamma}{u+1}} = f(u,\mu)$$

We use μ as bifurcation parameter as we vary μ with varying rate q.

We observe that $f(u,\mu) = 0 \Leftrightarrow$

$$u\mu = (b-u) e^{-\gamma/u+1} = h(u).$$

This equation cannot be solved explicitly for for u as a function of μ, but we can use graphical techniques for analyzing the bifurcation diagram



$\leftarrow$ Graph of $v = h(u)$ intersects $v = \mu u$ for various values of μ.

Thus for μ_1 (smallest μ is μ_1 here on figure) has largest u, and the bifurcation diagram is