Week 40, 2019 Mathematical Modelling

Supplementary notes, part 2, Krogstad

Dynamic population models

The logistic equation

The logistic equation models the population of individuals as a function of time. The model accounts for that a population cannot increase without bounds, and involves a given constant $K > 0$, signifying the maximal population possible (as long as the initial population is less than K). The equation is

$$\frac{dN^*}{dt^*} = rN^*(1 - N^*/K) , \quad (1)$$

where $N^*(t^*)$ is the population, and r ($[r] = \frac{1}{\text{time}}$) a growth factor. We see that if $N^* << K$, then $N^*(t) \approx N_0 e^{rt}$, thus $\frac{1}{r} = T$ is a natural timescale. Scaling (1) with $N^* = KN$, $t^* = \frac{1}{r}t$ we obtain

$$N' = N(1-N) \Rightarrow$$

$$\frac{dN}{N(1-N)} = dt \Leftrightarrow \left(\frac{1}{N} + \frac{1}{1-N}\right) dN = dt$$

$$\Rightarrow \ln\left|\frac{N}{1-N}\right| = t + \hat{c} \Rightarrow \frac{N}{1-N} = c e^{t} \Leftrightarrow N(t) = \frac{1}{1+Ae^{-t}}$$

If $N(0) = N_0$, we obtain

$$N(t) = \frac{N_0}{N_0 + (1-N_0)e^{-t}} = \frac{N_0}{e^{-t} + N_0(1-e^{-t})},$$

thus the denominator is always positive for $N_0 \geq 0$.

Also $\lim_{t\to\infty} N(t) = 1$ for any $N_0 > 0$.

Thus the equilibrium point $N_e = 0$ is unstable, and $N_e = 1$ is asymptotically stable.

<u>Whales and krill</u>

We consider a population model with two species, where one spicies (whales) feed on the other spicies. Also humans catch both species at rate proportional to each spicies population.

Let $N^*(t^*)$ and $H^*(t)$ be the number of krill and whales at time t^*. We consider the following model :

$$\frac{dN^{\infty}}{dt^{\infty}} = rN^{\infty}\left(1-\frac{N^{\infty}}{K_N}\right) - \alpha_2 N^{\infty}H^{\infty} - u_N F_N N^{\infty} \qquad (2)$$

$$\frac{dH^{\infty}}{dt^{\infty}} = q H^{\infty}\left(1-\frac{H^{\infty}}{\alpha N^{\infty}}\right) - u_H F_H H^{\infty} \quad , \qquad (3)$$

where $T_N = \frac{1}{r}$, $T_H = \frac{1}{q}$ are time scales for krill and whales as discussed in the logistic equation paragraph. We see that the whale "maximal" population is αN^{∞}. $\alpha_2 N^{\infty} H^{\infty}$ is the rate of krill beeing eaten by whale. u_N and u_H are the number of boats praying on krill and whale. Finally F_N and F_H are the fishing intensity of each boat. $[F_N]=[F_H] = \frac{1}{\text{time}}$. We scale the equations:

$$N^{\infty} = K_N N$$

$$H^{\infty} = \alpha K_N H$$

$$t^{\infty} = \frac{1}{q} t \qquad \text{(use the longest timescale)}$$

(2) is then

$$q \cancel{K_N} N' = r \cancel{K_N} N\left(1-\frac{K_N N}{K_N}\right) - \alpha_2 \alpha \cancel{K_N} K_N HN - u_N F_N \cancel{K_N} N$$

$$\Leftrightarrow \left(\frac{q}{r}\right) N' = N(1-N) - \left(\frac{\alpha_2 \alpha K_N}{r}\right) HN - \left(\frac{u_N F_N}{r}\right) N$$

$$\Leftrightarrow \quad \underline{\varepsilon N' = N(1-N) - \gamma H N - f_N N}$$

(3) gives

$$q \cancel{\alpha K_N} H' = \cancel{\alpha} K_N H q \left(1 - \frac{\alpha K_N H}{\alpha K_N N}\right) - u_H F_H \cancel{\alpha K_N} H$$

$$\Leftrightarrow \quad \underline{H' = H\left(1 - \frac{H}{N}\right) - f_H H}, \quad \text{where} \rightarrow$$

$$\varepsilon = \frac{q}{r}$$

$$\gamma = \frac{\alpha \alpha_2 K_N}{r}$$

$$f_N = \frac{F_N u_N}{r}$$

$$f_H = \frac{F_H u_H}{q}$$

<u>Stability</u> We consider equilibrium points of the dynamical system

$$\varepsilon N' = N(1-N) - \gamma H N - f_N N = N(1 - f_N - N - \gamma H)$$

$$H' = H\left(1 - \frac{H}{N}\right) - f_H H = H\left(1 - f_H - \frac{H}{N}\right).$$

Write $\vec{f}(N,H) = \begin{bmatrix} N(1 - f_N - N - \gamma H) \\ H\left(1 - f_H - \frac{H}{N}\right) \end{bmatrix}$

<u>Assume $H_e = 0$</u>, then $N_e = 0$ and $N_e = 1 - f_N$ give $(0,0)$ and $(1 - f_N, 0)$ as equilibrium points.

Let $H_e > 0$. Then

(4) $1 - f_N - N - \gamma H = 0$

(5) $1 - f_H - \frac{H}{N} = 0 \quad \Rightarrow \quad \underline{H = (1 - f_H) N}$

giving $1 - f_N - N - \gamma(1 - f_H) N = 0$

$$\Leftrightarrow \quad N = \frac{1 - f_N}{1 + \gamma(1 - f_H)}$$

Note that $1-f_H > 0$ and $1-f_N > 0$ since $H>0$ and $N>0$. Thus we have the equilibrium point

$$(N_e, H_e) = \left(\frac{1-f_N}{1+\gamma(1-f_H)}, \frac{(1-f_H)(1-f_N)}{1+\gamma(1-f_H)} \right). \quad (6)$$

To assess the stability properties of the equilibrium points, we calculate the Jacobi matrix of $\vec{f}(N,H)$

$$\vec{df} = \begin{bmatrix} 1-f_N-2N-\gamma H & -\gamma N \\ \frac{H^2}{N^2} & 1-f_H-2\frac{H}{N} \end{bmatrix}$$

For (N_e, H_e) given by (6) we have

$$\vec{df}(N_e, H_e) = \begin{bmatrix} -N_e & -\gamma N_e \\ (1-f_H)^2 & -(1-f_H) \end{bmatrix} = A$$

by using (4) and (5).

Since $\det A = N_e(1-f_H) + \gamma N_e (1-f_H)^2 > 0$ we either have two complex conjugate eigenvalues or two real eigenvalues of the same sign.

Now $\operatorname{trace} A = -(N_e + 1 - f_H) < 0$

Thus the real part of $\lambda = a \pm ib$ is negative or both real eigenvalues are negative.

Conclusion: (N_e, H_e) is asymptotically stable.

Consider now $(N_e, H_e) = (1-h_N, 0)$. Then

$$\vec{df}(N_e, H_e) = \begin{bmatrix} -(1-h_N) & -\gamma(1-h_N) \\ 0 & (1-h_H) \end{bmatrix} = A$$

Then $\det A = -(1-h_N)(1-h_H) < 0$. Thus the eigenvalues are real with different signs. Thus (N_e, H_e) is unstable.

Actually, we see that $\lambda_1 = -(1-h_N)$ and $\lambda_2 = (1-h_H)$ and the general solution to $\begin{bmatrix} \hat{N} \\ \hat{H} \end{bmatrix}' = A \begin{bmatrix} \hat{N} \\ \hat{H} \end{bmatrix}$

is: $$\begin{bmatrix} \tilde{N} \\ \tilde{H} \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-(1-h_N)t} + C_2 \begin{bmatrix} 1 \\ -\dfrac{(1-h_N+1-h_H)}{\gamma(1-h_N)} \end{bmatrix} e^{(1-h_H)t}$$

So $C_2 \neq 0$ (i.e. $\tilde{H}(0) \neq 0$) gives instability.

For $(N_e, H_e) = (0,0)$ the Jacobi matrix is not defined. For small $\tilde{N}$ and $\tilde{H}$ we have

$$\varepsilon \tilde{N}' \approx (1-f_N)\tilde{N} \quad \text{and}$$

$$\tilde{H}' \approx (1-f_H)\tilde{H}$$

signifying that $(0,0)$ is unstable.

It is possible to build an economic model on top of this model, where the profit of catching krill and whale is included. Also issues related to transport capacity etc. can be included in such a model.

Modelling based on conservation principles

Supplementary notes, part 3, Krogstad

We have already been using conservation of something (salt, water, amoebae, chemical, etc.) when deriving the governing equation(s) for the model. So far all models have either been spatially 0- or 1-dimensional. Classically we live in a 3D world (note string "theory" requires space to be 9- or 10-dimensional), and we will formulate "the universal conservation law" in 3D. Such requires some knowledge about vector-analysis such as flux surface integrals and the divergence theorem. However, in the following we only use the conceptual ideas without going into direct calculations.

Main concepts

The three players in a conservation law is

* conserved density φ
* flux density $\vec{J}$
* source/sink Q

Conserved density

We have a variable $\rho(x,y,z,t) = \rho(\vec{x},t)$ giving the density of some quantity φ. Thus $[\rho] = \frac{[\varphi]}{\text{volume}}$, and the amount of φ in a region $R \subset \mathbb{R}^3$ is

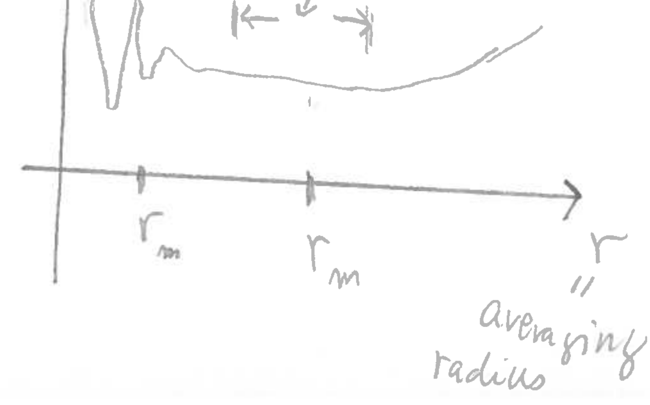
$$\varphi_R = \iiint_R \rho(x,y,z,t)\,dx\,dy\,dz = \iiint_R \rho(\vec{x},t)\,dV$$

Length scale; moving average

Consider a gas where r_m is length of mean free path. (r_m typically depends on pressure). On length scales $\approx r_m$ and less it is not proper to talk of the density of the gas. We really need to count the number of gas molecules a region with a length scale $r_a \gg r_m$, and define the the density $\rho(\vec{x}_0, t)$ as a moving average. That is, let $B_{\vec{x}_0} = \{\vec{x} \in \mathbb{R}^3 : \|\vec{x} - \vec{x}_0\| < r_a\}$, be the ball centered at $\vec{x}_0$ with radius r_a. Then the (molar) density of the gas is

$$\rho(\vec{x}_0, t) = \frac{\text{number of molecules in } B_{\vec{x}_0}}{\text{volume of } B_{\vec{x}_0}}$$

So, when $\vec{x}_0$ varies, we have defined $\rho(\vec{x}_0, t)$ as a moving average.



We say that we have separation of scales if $r_m \ll r_a \ll r_{macro}$

where r_{macro} is the lengthscale that $\rho(\vec{x}_0, t)$ change significantly.

Examples of conserved densities

Quantity	density	
mass	ρ	
momentum	$\rho\vec{v}$	, $\vec{v}$ velocity
thermal energy	ρe	, e spesific thermal energy density.
charge	q	

Flux

The flux φ_D of a quantity φ over a surface region $D \subset \mathbb{R}^3$ is the amount of φ crossing this surface per time. (Note, we must define the sign of φ_D by choosing a positive direction for what we mean by crossing.) That is, D must be an oriented surface).

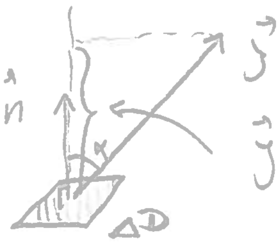
The flux density is a (in general time dependent) vectorfield $\vec{J}(\vec{x}, t)$, such that

For a small oriented surface ΔD with unit normal vector $\vec{n}$ (small, such that $\vec{j}$ is approximately constant on ΔD), the flux over ΔD is

$$\varphi_{\Delta D} = \vec{j}\cdot\vec{n}\,\Delta S\,,$$

where ΔS is the area of ΔD.

$$[\vec{j}] = \frac{[\varphi]}{\text{time}\cdot\text{area}}$$



$$\vec{j}\cdot\vec{n} = |\vec{j}|\cos\alpha$$

Dividing an oriented surface D into many small surfaces ΔD, and summing the flux over each of the ΔD surfaces gives approximately the flux of φ over D. That is

$$\varphi_D \approx \sum_{\Delta D \text{ over } D} \varphi_{\Delta D} = \sum_{\text{over } D} \vec{j}\cdot\vec{n}\,\Delta S$$

$$\xrightarrow{\text{"}\Delta S\to 0\text{"}} \iint_D \vec{j}\cdot\vec{n}\,dS = \varphi_D$$

Example: Let $D = \{(x,y,0)\in\mathbb{R}^3 : 0<x<x_0,\ 0<y<y_0\}$ (rectangle in the xy-plane) with $\vec{n} = [0,0,1]$

Let $\vec{j} = [ax,\ b(x^2+z^2),\ cxyt+dz]$ (any vectorfield), a, b, c, d const.

Then $\vec{j}(x,y,0,t)\cdot\vec{n} = ctxy$. Thus,

$$\varphi_D = \iint_D \vec{J}\cdot\vec{n}\,dS = \int_0^{y_0}\int_0^{x_0} ctxy\,dx\,dy = \left(\frac{c x_0^2 y_0^2}{4}\right)t$$

Divergence theorem

Let $R \subset \mathbb{R}^3$ be a bounded region with piecewise smooth surface ∂R. Assume the components of the vectorfield $\vec{J}$ are continuously differentiable in the whole of R, $\vec{n}$ points out of R. Then

$$\iint_{\partial R} \vec{J}\cdot\vec{n}\,dS = \iiint_R (\vec{\nabla}\cdot\vec{J})\,dV\,,$$

where $\vec{\nabla}\cdot\vec{J} = \vec{\nabla}\cdot\begin{bmatrix} J_1 \\ J_2 \\ J_3 \end{bmatrix} = \frac{\partial J_1}{\partial x} + \frac{\partial J_2}{\partial y} + \frac{\partial J_3}{\partial z}$

is the divergence of $\vec{J}$.

We will use (when possible) this theorem later when expressing the conservation law on differential form.

Sources/sinks

A (positive) source is an external supply of φ to the model we study. A sink (= negative source) drains our model of φ, reducing the total amount of φ in the model. We operate with a source density $Q(\vec{x},t)$, $[Q] = \frac{[\varphi]}{\text{time}\cdot\text{volume}}$, such that

$$\iiint_R Q(\vec{x},t)\,dV = \text{amount of } \varphi \text{ entering}$$

our model per time from the region R.

$Q > 0$ we have a source (or positive source)

$Q < 0$ we have a sink (or negative source)

<u>Regular sources</u>

When $Q(\vec{x},t)$ is a <u>function</u> of $\vec{x}$ we have

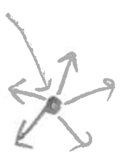
$$\lim_{r\to 0} \iiint_{B_{\vec{x}_0}(r)} Q(\vec{x},t)\,dV = 0$$

, where $B_{\vec{x}_0}(r)$ is a ball with radius r. Then Q is a regular source.

<u>Point sources</u>

Often it is conveinient and appropriate to model a source as a point source at $\vec{x}_0$, and define

point source



$$\iiint_R Q\,dV = \begin{cases} q & \text{if } x_0 \in R \\ 0 & \text{if } x_0 \notin R \end{cases}$$

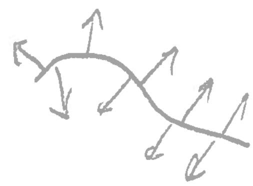
We formally write $Q(\vec{x},t) = q\,\delta_{\vec{x}_0}(\vec{x})$, where $\delta_{\vec{x}_0}(\vec{x})$ is the so-called Dirac delta-function. $\delta_{\vec{x}_0}(\vec{x})$ is mathematically a distribution (a linear functional on a suitable function space).

Line sources

If $C \in \mathbb{R}^3$ is a curve, and the source density Q satisfies

$$\iiint_R Q\,dV = \int_{C\cap R} Q\,ds = \qquad (C\cap R \text{ is the part of } C \text{ inside } R)$$

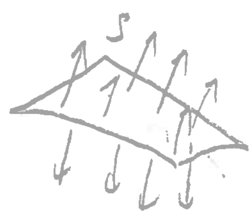
Q is a line source



Surface source

Likewise, if $\iiint_R Q\,dV = \iint_{R\cap S} Q\,dS$

we have a surface source, where $S \subset \mathbb{R}^3$ is a surface (not oriented).



Universal conservation law

Let $\rho(\vec{x},t)$ be the density of the conserved quantity φ, $\vec{J}(\vec{x},t)$ the flux density, and $Q(\vec{x},t)$ the source density. Let $R \subset \mathbb{R}^3$ be any region (subset of the domain of our model). Then the amount of φ in R is

$$\iiint_R \rho \, dV .$$

The rate of φ flowing out over the surface ∂R is

$$\iint_{\partial R} \vec{j} \cdot \vec{n} \, dS \, ,$$ where $\vec{n}$ points out of R.

$\iiint_R Q \, dV$ is the rate of φ entering R from external sources.

Thus,

$$\frac{d}{dt} \iiint_R \rho \, dV = \underbrace{- \iint_{\partial R} \vec{j} \cdot \vec{n} \, dS}_{\text{rate of } \varphi \text{ flowing into } R \text{ over } \partial R} + \iiint_R Q \, dV$$

$\Leftrightarrow$

$$\boxed{\frac{d}{dt} \iiint_R \rho \, dV + \iint_{\partial R} \vec{j} \cdot \vec{n} \, dS = \iiint_R Q \, dV} \qquad (6)$$

which is the universial conservation law on integral form.

When $\vec{j}$ has continuously differentiable components, we can rewrite (6) somewhat. (Note that $\vec{j}$ may not have this property for a number of practical and realistic physical models). As long as R has piecewise smooth surface we can write

$$\iint_{\partial R} \vec{J}\cdot\vec{n}\, dS = \iiint_{R} \vec{\nabla}\cdot\vec{J}\, dV,$$

and we obtain

$$\frac{d}{dt}\iiint_{V} \rho\, dV + \iiint_{V} \vec{\nabla}\cdot\vec{J}\, dV - \iiint_{V} Q\, dV = 0.$$

Furthermore, under very general conditions we have

$$\frac{d}{dt}\iiint_{R} \rho\, dV = \iiint_{R} \frac{\partial \rho}{\partial t}\, dV,$$ meaning that

we can write

$$\iiint_{R} \left(\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} - Q \right) dV = 0 \quad (7)$$

Assume the integrand is continuous. Since R is an arbitrary domain, and (7) always holds we can conclude that

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} - Q = 0} \quad (8)$$

which is the <u>universal conservation equation on differential form</u>.

Example

Consider temperature $T(\vec{x},t)$ in $\mathbb{R}^3$ with no sources. The thermal energy density is

$$c \rho T = e$$

where c is specific heat capacity and ρ mass density of the medium. The flux density of of thermal energy is given by Fourier's law

$$\vec{J} = -k \vec{\nabla} T$$

where k is thermal conductivity. Energy conservation on differential form is then

$$\frac{\partial}{\partial t}(c \rho T) + \frac{\partial}{\partial x}(-k \vec{\nabla} T) = 0 .$$

If the medium is homogeneous, then ρ, c, k are constant, and we obtain

$$\frac{\partial T}{\partial t} = \left(\frac{k}{c\rho}\right) \nabla^2 T$$

$$\Leftrightarrow \frac{\partial T}{\partial t} = \left(\frac{k}{c\rho}\right)\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right)$$

whic is a (linear) partial differential equation stating that thermal energy is conserved,