TMA4195 Mathematical Modelling 2019, Week 41

Conservation law in 1D

The density of the conserved entity φ is ρ; i.e $[\rho] = \frac{[\varphi]}{\text{volume}}$

The flux density $\vec{j}$, $[\vec{j}] = \frac{[\varphi]}{\text{time}\cdot\text{area}}$, defines the flux through an oriented surface D by

$$\iint_D \vec{j}\cdot\vec{n}\,dS$$

where $\vec{n}$ is the unit normal vector of the oriented surface.

Q is the source density, meaning that $\iiint_R Q\,dV$ is the rate φ entering the model (source) from the domain R. $Q<0$ means the source is a sink.

Conservation of φ in any region R is

$$\frac{d}{dt}\iiint_R \rho\,dV + \iint_{\partial R} \vec{j}\cdot\vec{n}\,dS = \iiint_R Q\,dV, \qquad (1)$$

where $\vec{n}$ points out of R. Under general conditions (1) is equivalent to

$$\frac{d\rho}{dt} + \vec{\nabla}\cdot\vec{j} = Q \qquad (2)$$

In 1 spatial dimension, (1) and (2) reduces to

$$\frac{d}{dt}\int_a^b \rho(t,x)\,dx + j(t,b) - j(t,a) = \int_a^b Q(t,x)\,dx \qquad (3)$$

and

$$\frac{d\rho}{dt} + \frac{d}{dx} J = Q \qquad (4)$$

In 1-D we have $[\rho] = \frac{[\varphi]}{\text{length}}$, $[J] = \frac{[\varphi]}{\text{time}}$, $[Q] = \frac{[\varphi]}{\text{length}\cdot\text{time}}$

If $J = J(t,x,\rho)$, then (4) becomes

$$\left(\frac{d\rho}{dt}\right) + \frac{dJ}{dx} + \frac{dJ}{d\rho}\left(\frac{d\rho}{dx}\right) = Q \qquad (5)$$

(J also has x-dependence through $\rho(t,x)$).

Method of characteristics

(5) fits into the class of partial differential equations called quasi-linear 1. order equations

$$u_t + a(t,x,u)\, u_x = b(t,x,u) \qquad (6)$$

where a and b are given functions.
In addition, we assume the initial conditions are given by $u(0,x) = u_0(x)$. (7)

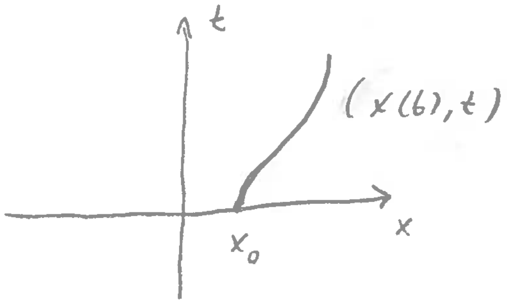
The idea is to transform the initial value problem (6)+(7) to a system of ordinary differential equations.

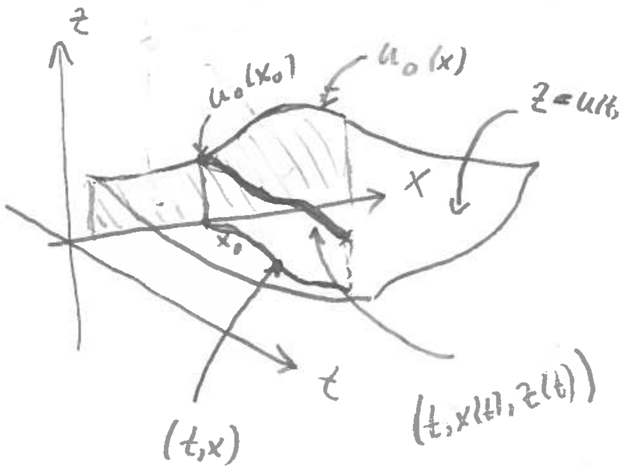
We find that the left hand side of (6) is the time derivative of

$$z(t) = u(t, x(t)) \qquad (u(t,x) \text{ is the solution of } (6)+(7))$$

if $x'(t) = a(t,x,z)$.

Then $z'(t) = b(t,x,z)$ = <u>right hand side of (6)</u>. ③





So, for each x_0 we seek a curve $(t, x(t), z(t))$, $t \geq 0$, starting at

$$(0, x(0), z(0)) = (0, x_0, u_0(x_0))$$ and satisfying

$$x' = a(t,x,z)$$
$$z' = b(t,x,z)$$

If we can solve this set of equations giving

$$x = x(t,x_0)$$
$$z = z(t,x_0)$$

(use x_0 in the notation where the curve starts),

then we can in principle solve for x_0 in

$$x = x(t,x_0)$$

giving $x_0 = x_0(t,x)$ (i.e. $x_0(t,x)$ gives the start point $(0, x_0(x,t))$ for the curve passing through (t,x)).

Finally, since $z(t) = u(t, x(t))$, we have

$$u(x,t) = z(t, x_0(t,x))$$ as the solution.

The curve $(t, x(t; x_0))$, $t \geq 0$ is called a <u>characteristic</u>.

Example

$$u_t + 3u_x = u^2 \quad , \quad u(0,x) = u_0(x)$$

Then $a(t,x,u) = 3$ and $b(t,x,u) = u^2$. Thus

$$x' = 3 \quad , \quad x(0) = x_0$$
$$z' = z^2 \quad , \quad z(0) = u_0(x_0)$$

Gives $x = 3t + x_0 \Leftrightarrow \underset{\sim\sim\sim}{x_0 = x - 3t}$, while

$$z' = z^2 \Leftrightarrow \frac{dz}{z^2} = dt \Rightarrow -\frac{1}{z} = t - c \ , \ c \in \mathbb{R} ,$$

$$\Leftrightarrow \underset{\sim\sim\sim}{z = \frac{1}{c-t}} \qquad z(0) = u_0(x_0) \text{ gives } c = \frac{1}{u_0(x_0)}$$

$$\Rightarrow z = \frac{1}{\frac{1}{u_0(x_0)} - t} = \frac{u_0(x_0)}{1 - t u_0(x_0)} .$$

Finally $u(t,x) = z(t, x_0(t,x)) = \underline{\underline{\dfrac{u_0(x-3t)}{1 - t u_0(x-3t)}}}$

The Riemann problem

We will assume now that in (6)

$$\underset{\sim\sim\sim}{a = j(u)} \ , \ \underset{\sim\sim}{b = 0}$$

Let $J'(u) = j(u)$. Then (6) is

$$u_t + (J(u))_x = 0 \Leftrightarrow u_t + J'(u) u_x = 0$$

which is a conservation law with flux density $J(u)$.

A Riemann problem is an intial value problem of the form

$$\left.\begin{aligned} & u_t + j(u) u_x = 0 \\ & u_0(x) = u(0,x) = \begin{cases} u_- & \text{for } x < 0 \\ u_+ & \text{for } x > 0 \end{cases} \end{aligned}\right\} \quad (8)$$

The method of characteristics give

$$x' = j(z) \qquad x(0) = X_0$$

$$z' = 0 \qquad z(0) = u_0(x_0)$$

Thus, $z(t) = z(0) = u_0(x_0)$ for $t > 0$

$\Rightarrow j(z) = j(u_0(x_0)) = \text{const.} \Rightarrow$

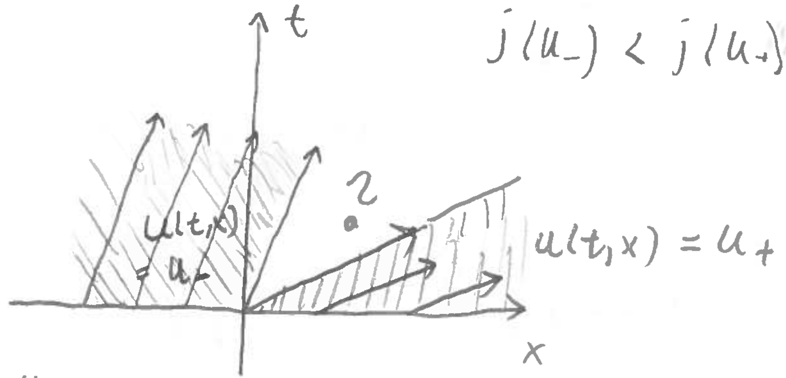
$$x = j(u_0(x_0)) t + x_0$$

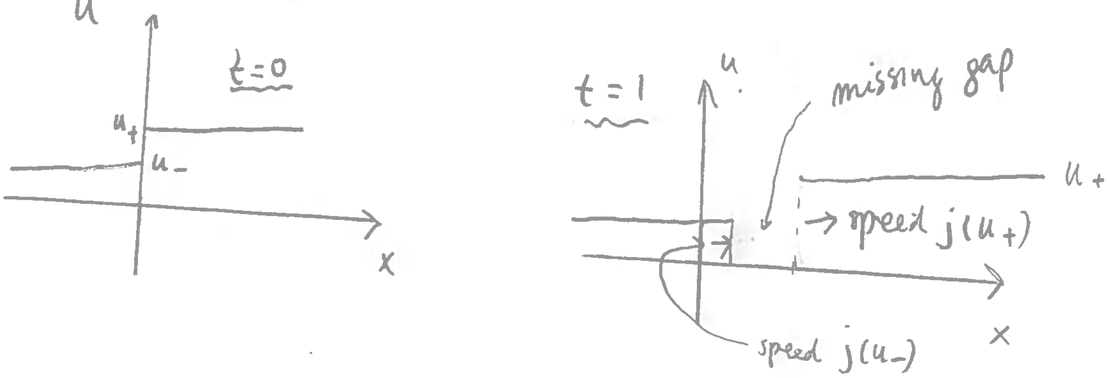
We see that the characteristics are straight lines, and on this straight line u is constant and equal to $u_0(x_0)$.

Rarefaction, $j(u_-) < j(u_+)$

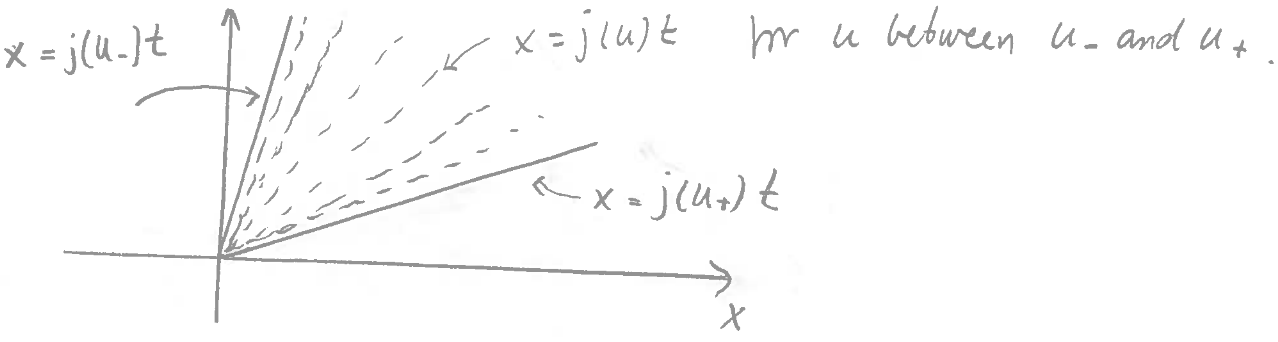
Since $j(u_0(x_0)) = \frac{dx}{dt}$ is the speed the constant value $u_0(x_0)$ moves in (t,x) plane, $j(u_-) < j(u_+)$ means that characteristics "move faster" for $x > 0$ than the do for $x < 0$. It is very usefull to draw characteristics in the (t,x) plane:

$j(u_-) < j(u_+)$





For this case $j(u_-) < j(u_+)$ we see that we get gap due to different speed. This gap is filled with a so-called <u>rarefaction wave</u> connecting u_- part of the solution to the u_+ part of the solution as long as $j(u)$ increases continuously monotonically as u moves from u_- to u_+.



So, in the rarefaction wave, each value u between u_- and u_+ moves with a speed $j(u)$ along the line $x = j(u)t$.

Therefore $j(u) = x/t \Leftrightarrow u = j^{-1}(\frac{x}{t})$ gives the solution for $j(u_-)t < x < j(u_+)t$.

We can also demonstrate that $u(x,t) = \eta(\frac{x}{t})$ solves the equation for $\eta(s) = j^{-1}(s)$. Insert $u = \eta(\frac{x}{t})$ in

$u_t + j(u)u_x = 0$ to obtain.

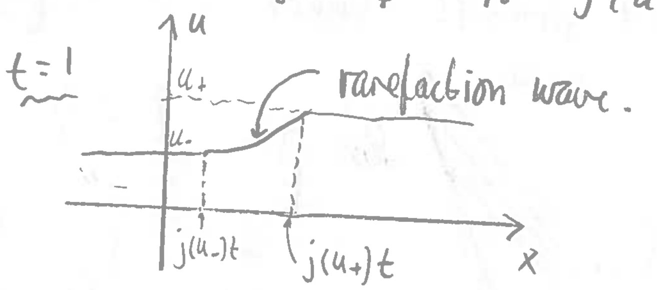
$$\eta'(\tfrac{x}{t})(-\tfrac{x}{t^2}) + j(\eta(\tfrac{x}{t}))\,\eta'(\tfrac{x}{t})\,\tfrac{1}{t} = 0$$

$$\Leftrightarrow \eta'(\tfrac{x}{t})\,\tfrac{1}{t}\left[-\tfrac{x}{t} + j(\eta(\tfrac{x}{t}))\right] = 0$$

Thus $j(\eta(\frac{x}{t})) = \frac{x}{t} \Leftrightarrow \eta(\frac{x}{t}) = j^{-1}(\frac{x}{t})$.

To sum up: If $j(u_-) < j(u_+)$ and j increases monotonically as u passes from u_- to u_+, then

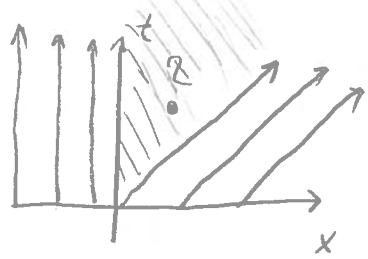
$$u(t,x) = \begin{cases} u_- & \text{for } x < j(u_-)t \\ j^{-1}(\frac{x}{t}) & \text{for } j(u_-)t < x < j(u_+)t \\ u_+ & \text{for } j(u_+)t < x \end{cases}$$



Example $u_t + u^2 u_x = 0$ $\quad u_0(x) = \begin{cases} 0 & \text{for } x<0 \\ 1 & \text{for } x>0 \end{cases}$

Characteristics $x = u_0^2(x_0)t + x_0$

$$= \begin{cases} x_0 & \text{for } x_0 < 0 \\ t + x_0 & \text{for } x_0 > 0 \end{cases}$$



Put $u(x,t) = \eta(\frac{x}{t})$ in the equation to obtain $\eta'\frac{1}{t}[-\frac{x}{t} + u^2] = 0$

$\Rightarrow \; u^2 = \frac{x}{t} \;\Rightarrow\; u = \sqrt{\frac{x}{t}}$ (want the positive sign since $u = \sqrt{\frac{x}{t}}$

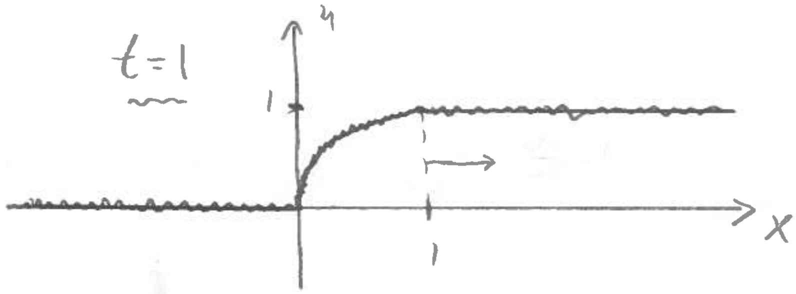
connects $u=0$ with $u=1$ as x goes from 0 to t.

The solution to the Riemann problem is consequently

$$u(x,t) = \begin{cases} 0 & \text{for } x<0 \\ \sqrt{x/t} & \text{for } 0<x<t \\ 1 & \text{for } x>t \end{cases}$$

Note that the solution is not differentiable for $x=0$ and $x=t$. However, it is continuous, even though $u_0(x)$ was not

$t=1$



Shocks

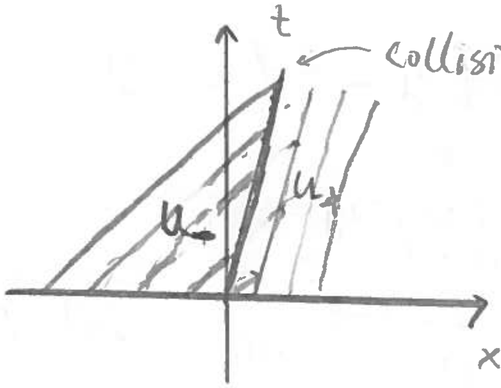
As mentioned, when the speeds of the characteristics increase as x_0 (start value for characteristic) increases, the characteristics will not cross oneanother.

When this is not the case, the solution develops shocks, that is, discontinuities. Such discontinuitis travel with a certain speed, that may depend on time. Let us consider the Riemann problem

$$u_t + j(u)\,u_x = 0$$

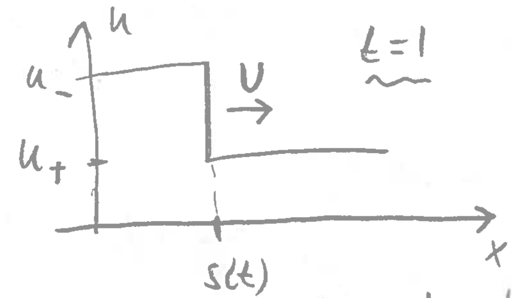
$$u(0,x) = \begin{cases} u_-, & x<0 \\ u_+, & x>0 \end{cases}$$

where this time $j(u_-) > j(u_+)$



← collision of characteristics at $x = j(u_+)t$

This means that characteristics starting from $x_0 < 0$ will cross the characteristic $x = j(u_+)t$ starting at $x_0 = 0$, where $u = u_-$ on one side and $u = u_+$ on the other side of the shock. Thus, we picture a solution of the form



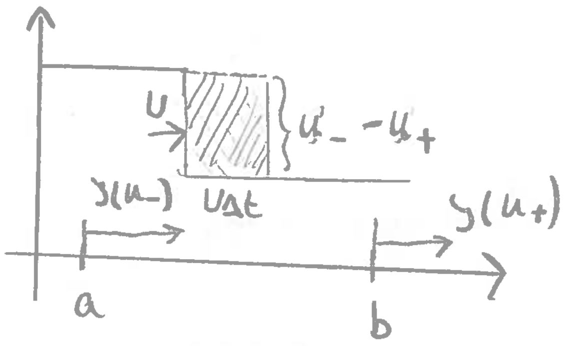
where the discontinuity travels at some speed U.

The speed U is related to how big the discontinuity is, and what flux densities are on each side of the discontinuity.

Let $J'(u) = j(u)$, so $J(u)$ is the flux density.

For an interval $[a,b]$, conservation in integral form is

$$\frac{d}{dt}\int_a^b u\,dx = J(u(t,a)) - J(u(t,b))$$



During $[t, t+\Delta t]$, the net increase of the conserved entity in $[a,b]$ is $\Delta t\,(\mathcal{J}(u_-) - \mathcal{J}(u_+))$

The net increase of this entity due to moving discontinuity is area of shaded region in the figure

$$(u_- - u_+)\,U\Delta t \qquad (U \text{ speed of discont.})$$

Thus $(u_- - u_+)\,U = \mathcal{J}(u_-) - \mathcal{J}(u_+)$

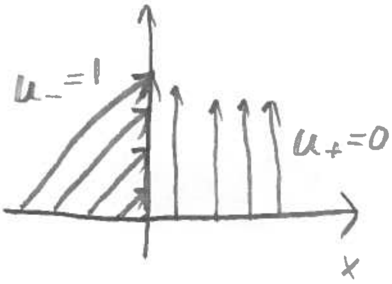
$$\Leftrightarrow \quad \boxed{U = \frac{\mathcal{J}(u_+) - \mathcal{J}(u_-)}{u_+ - u_-}} \qquad (9)$$

which is the Rankine-Hugoniot condition, defining the speed of the discontinuity or shock.

Example $\quad u_t + u^2 u_x = 0$

$$u(0,x) = \begin{cases} 1 & x<0 \\ 0 & x>0 \end{cases}$$

Characteristics: $\quad x(t) = \begin{cases} x_0 + t & \text{for } x_0 < 0 \\ x_0 & \text{for } x_0 > 0 \end{cases}$



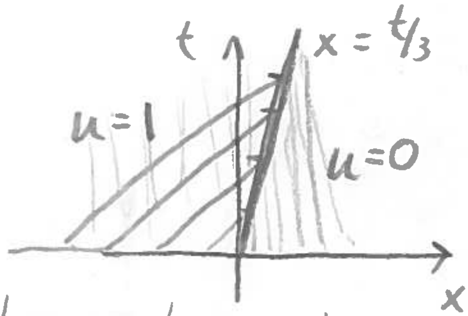
Shock develops at $x=0$.

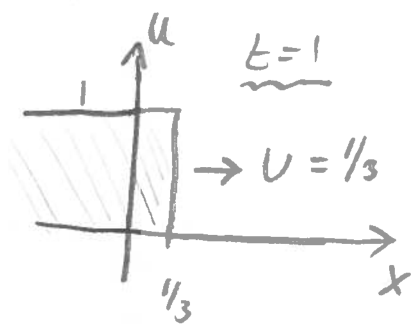
Since $J'(u) = u^2 = j(u)$

we have $J(u) = \frac{1}{3}u^3$ (integration constant not important)

Thus (9) gives $U = \dfrac{\frac{1}{3}0^3 - \frac{1}{3}1^3}{0-1} = \frac{1}{3}$

Thus we have a shock for $x = s(t) = \frac{1}{3}t$





Comments on next example

In the next example (next page), we will look at a "double" Riemann problem where u_- varies in time after a while (when $t > t_m$, where t_m can be calculated). Consequently, the shock speed will not be constant in time.

The solution at a given time is illustrated with an Excel sheet (see link where you found these notes). Be sure to see the connection between the solution for various t and the analysis on the next page.

Example "Double" Riemann problem

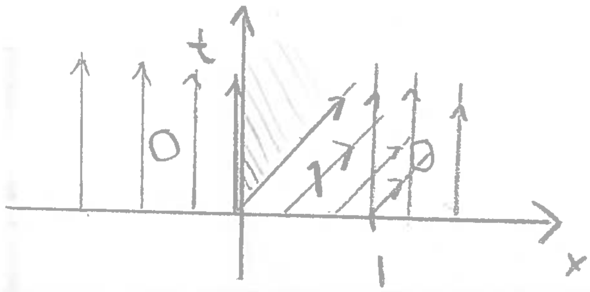
For $p>0$ consider

(1) $u_t + u^p u_x = 0 \quad , x \in \mathbb{R}, t \geq 0$

$$u(0,x) = \begin{cases} 0 & \text{for } x<0 \text{ and } x>1 \\ 1 & \text{for } 0<x<1 \end{cases}$$

Characteristics are

$$x(t) = \begin{cases} x_0 & \text{for } x_0<0 \text{ and } x_0>1 \\ x_0+t & \text{for } 0<x_0<1 \end{cases}$$



The characteristics tell us 3 things:

1) For $x<0$ $\quad u(t,x)=0$ for all $t>0$

2) We have a rarefaction wave for $0<x<t$

3) A shock develops at $x=1$

We know that $u(t,x) = \varphi(\frac{x}{t})$ for the rarefaction wave.

Inserted into (1) gives

$$\varphi'(\tfrac{x}{t})\frac{1}{t}\left[-\frac{x}{t} + \varphi^p(\tfrac{x}{t})\right] = 0$$

$$\Rightarrow \varphi(\tfrac{x}{t}) = (\tfrac{x}{t})^{1/p} = u(t,x) \text{ for } 0<x<t.$$

(13)

To find the position of the shock at time t, $x = s(t)$, we know that

$$U = s'(t) = \frac{\mathcal{J}(u_+) - \mathcal{J}(u_-)}{u_+ - u_-},$$

where $\mathcal{J}(u) = \frac{1}{p+1} u^{p+1}$, since $\mathcal{J}'(u) = u^p$.

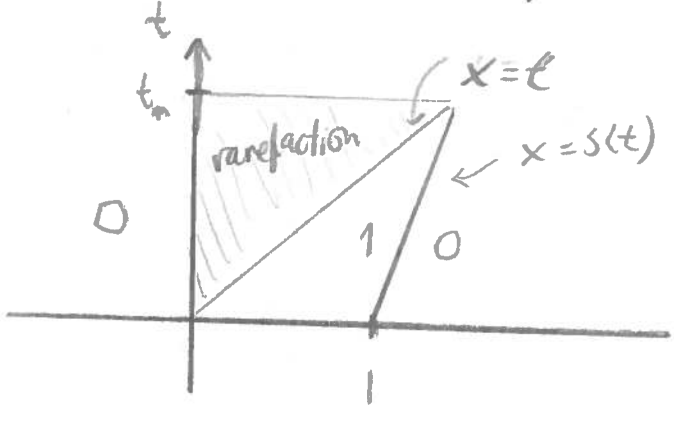
We have (for sufficiently small t)

$$u_- = 1, \quad u_+ = 0,$$

Thus $U = s'(t) = \frac{1}{p+1} \frac{0^{p+1} - 1^{p+1}}{0 - 1} = \frac{1}{p+1}$.

Since $s(0) = 1$ we get

$$s(t) = 1 + \frac{t}{1+p}$$



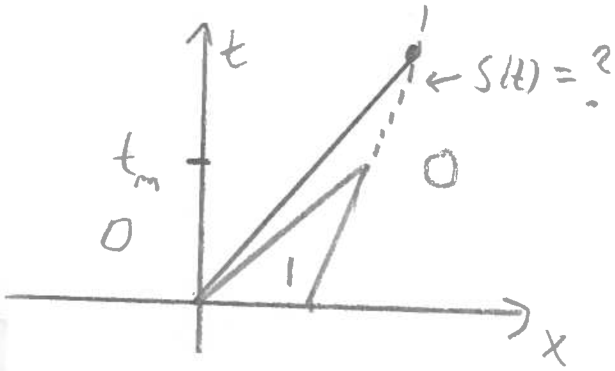
We see that $u_- = 1$ and $u_+ = 0$ on each side of the shock as long as $t < t_m$, where $t = t_m$ gives the intersection of $x = t$ and $x = s(t)$, i.e.

$$t = s(t) \Leftrightarrow t = 1 + \frac{t}{1+p} \Leftrightarrow t = t_m = \underline{\frac{1+p}{p}}$$

Thus the solution for $\underline{t < t_m}$ is

$$u(t,x) = \begin{cases} 0 & , x < 0 \\ \left(\frac{x}{t}\right)^{1/p} & , 0 < x < t \\ 1 & , t < x < s(t) \\ 0 & , x > s(t) \end{cases}$$

For $t > t_m$ we see that no longer is $u_- = 1$



We see that u_- is given by $u(t,x) = u(t,s(t))$ when the shock is at position $x = s(t)$, where

$u(t,x) = \left(\frac{x}{t}\right)^{1/p}$. i.e. $u_- = \left(\frac{s(t)}{t}\right)^{1/p}$ and $u_+ = 0$.

Also the speed of the shock is $U = s'(t)$.

Thus, the Rankine-Hugoniot condition is

$$U = s' = \frac{J(u_+) - J(u_-)}{u_+ - u_-} \iff$$

$$s' = \frac{1}{p+1} \frac{0 - \left(\left(\frac{s}{t}\right)^{1/p}\right)^{1+p}}{0 - \left(\frac{s}{t}\right)^{1/p}} = \frac{1}{p+1} \frac{s}{t} . \quad (2)$$

This is a separable 1-order differential equation for $s(t)$, with initial condition

$$s(t_m) = t_m .$$

Solving (2) we obtain

$$s(t) = c\, t^{\frac{1}{1+p}} , \quad c \in \mathbb{R} .$$

$t_m = \frac{1+p}{p}$, and $s(t_m) = t_m \Leftrightarrow$

$$c \left(\frac{1+p}{p}\right)^{\frac{1}{1+p}} = \left(\frac{1+p}{p}\right)^{1} \Leftrightarrow c = \left(\frac{1+p}{p}\right)^{1-\frac{1}{1+p}}$$

$\Leftrightarrow c = \left(\frac{1+p}{p}\right)^{\frac{p}{1+p}}$. Thus

$$\underline{s(t) = \left(\frac{1+p}{p}\right)^{\frac{p}{1+p}} t^{\frac{1}{1+p}}} \quad \text{for } t > \frac{1+p}{p} .$$

The solution for $\underline{t > \frac{1+p}{p}}$ is consequently

$$u(t,x) = \begin{cases} 0 , & x < 0 \\ \left(\frac{x}{t}\right)^{1/p} , & 0 < x < s(t) \\ 0 , & x > s(t) \end{cases}$$

.