Mathematical Modelling TMA4195, Week 42, 2019

Traffic Modelling (Krogstad, Supp. notes 3)

We model the traffic density u^* along a 1D road, where

$$u^* = \frac{\text{number of cars}}{\text{length}},$$

and of course, u^* varies with position and time.

We assume there is a maximal density of cars, u_{max}, and that the velocity of a car is a function of u^* at the car's position. It is reasonable to assume that the velocity $v^*(u^*)$ is decreasing as u^* increases, and that $v^*(u_{max}) = 0$ and $v^*(0) = v_{max}$, where v_{max} is the maximal velocity. The simplest such function is

$$v^*(u^*) = v_{max}\left(1 - \frac{u^*}{u_{max}}\right).$$

The flux density is consequently

$$j^* = u^* v^*(u^*) = v_{max}\, u^*\left(1 - \frac{u^*}{u_{max}}\right).$$

Conservation of cars (no cars leave or enter the road) is then

$$\frac{\partial u^*}{\partial t^*} + \frac{\partial}{\partial x^*} j^*(u^*) = 0$$

$$\Leftrightarrow \quad \frac{du^\infty}{dt^\infty} + v_{max}\left(1 - \frac{2u^\infty}{u_{max}}\right)\frac{du^\infty}{dx^\infty} = 0$$

Scaling: Set $u^\infty = u_{max} u \qquad (u \in [0,1])$

$$\left.\begin{array}{l} x^\infty = Lx \\ t^\infty = Tt \end{array}\right\}$$ where L and T are to be determined by balancing the equation.

This gives
$$\frac{u_{max}}{T} u_t + v_{max}(1-2u)\frac{u_{max}}{L} u_x = 0$$

$$\Leftrightarrow \quad u_t + \left(\frac{T v_{max}}{L}\right)(1-2u)u_x = 0$$

So, as long as $\frac{L}{T} = v_{max}$ we have

$$\boxed{u_t + (1-2u)u_x = 0}$$

(the initial condition may include a "typical" lengthscale a, so then $L = a$ and $T = \frac{a}{v_{max}}$)

Example 1 Red to green light at $x=0$ at time $t=0$:

Imagine a traffic light changes from red to green at $x=0$ at time $t=0$, so that the initial condition is

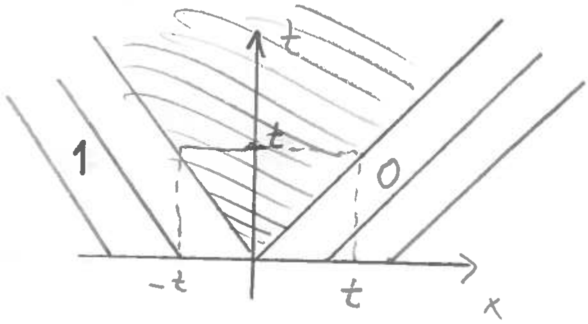
$$u_0(x) = \begin{cases} 1 & \text{for } x<0 \\ 0 & \text{for } x>0 \end{cases}$$

Characteristics starting at $(0, x_0)$ in the (t,x)-plane are

$$x(t) = x_0 + j(u_0(x_0))t$$

where $\quad j(u) = J'(u) = 1-2u$.

For $u_0 = 1$ we have $j(1) = -1$, and for $u_0 = 0$, $j(0) = 1$.
Thus, the characteristics are



$$x_0 < 0 : \quad x(t) = x_0 - t$$

$$x_0 > 0 : \quad x(t) = x_0 + t$$

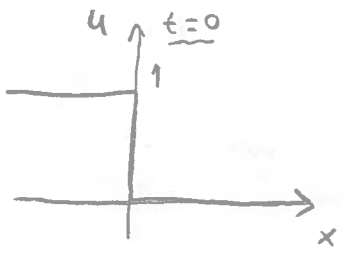
Thus, for $\; -t < x < t$ we have a rarefaction wave, and the solution is given by (note $j(u)$ is monotonous)

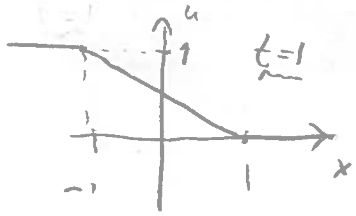
$$\frac{x}{t} = j(u) = 1 - 2u$$

$$\Leftrightarrow \quad u = \frac{1}{2}\left(1 - \frac{x}{t}\right)$$

Thus the solution is

$$u(t,x) = \begin{cases} 1 & \text{for } x < -t \\ \frac{1}{2}\left(1 - \frac{x}{t}\right) & \text{for } -t < x < t \\ 0 & \text{for } x > t \end{cases}$$





At $t=1$, the car in line at $x=-1$ starts driving, and the car at $x=0$ at $t=0$ has reached $x=1$ at $t=1$.

Let's find the position $x = y(t)$ of the car starting at $x=-1$ at $t=1$. We know that the velocity is given by

$$v(u(t, y(t))) = 1 - u(t, y(t))$$

Since the car is situated in the rarefaction part at all times $t > 1$ we have

$$v(u(t,y)) = 1 - \frac{1}{2}\left(1 - \frac{y}{t}\right) = \frac{1}{2} + \frac{1}{2}\frac{y}{t} \quad \text{for } t > 1.$$

Since $y' = v$ we get the linear 1. order equation

$$y' = \frac{1}{2} + \frac{1}{2}\frac{y}{t}$$

with general solution $y(t) = t + C t^{1/2}$.

$y(1) = -1$ gives then $\underline{y(t) = t - 2t^{1/2}}$

The cars velocity is $y'(t) = 1 - \frac{1}{t^{1/2}}$, increasing from 0 to 1, with acceleration $y''(t) = \frac{1}{2}\frac{1}{t\sqrt{t}}$.

Note that the position of the first car is $y_1(t) = t$, thus the distance between $y(t)$ and $y_1(t)$ is

$$y_1(t) - y(t) = 2t^{1/2}, \quad \text{for } t > 1.$$

and we see how this distance increases with time.

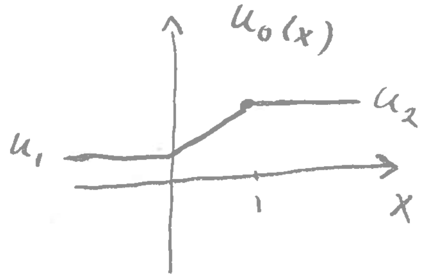
(or as they phrase it in sailing competitions: "a good start is half the victory")

<u>Example 2</u> <u>"Traffic up ahead"</u>

We consider the case where we have increasing car density from $x = 0$ to $x = a$. Now we have a length scale, so we let $L = a$ ($x^* = ax$, $t^* = \frac{a}{v_{max}} t$). The initial condition is

$$u_0(x) = \begin{cases} u_1 & \text{for } x<0 \\ u_1+(u_2-u_1)x & \text{for } 0<x<1 \\ u_2 & \text{for } x>1 \end{cases}$$

where $u_1 < u_2$.



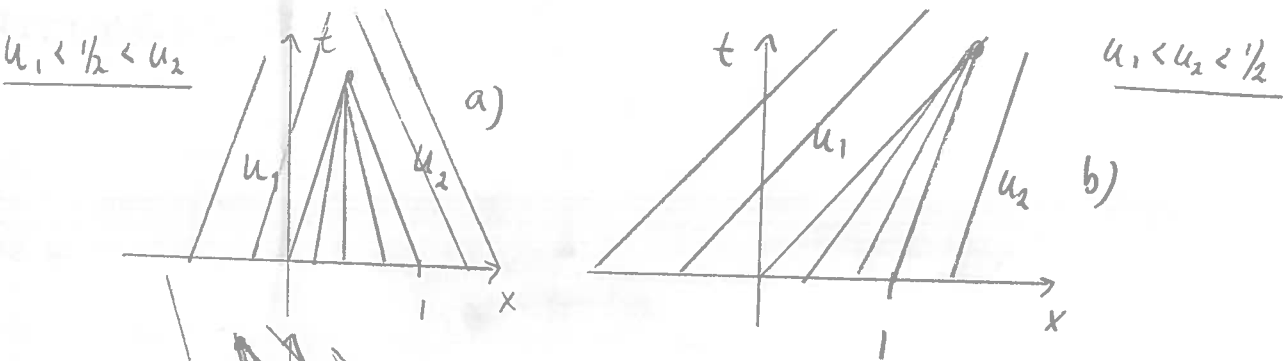
Characteristics:

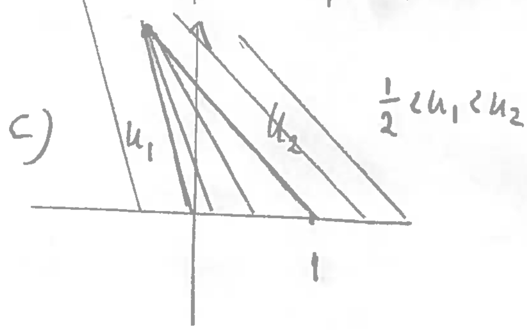
$$x(t) = x_0 + j(u_0(x_0))t$$

For $x_0<0$ $\quad x(t) = x_0 + (1-2u_1)t$

$0<x_0<1$ $\quad x(t) = x_0 + (1-2(u_1+(u_2-u_1)x_0))t$

$x_0>1$ $\quad x(t) = x_0 + (1-2u_2)t$





Characteristics for various values of u_1 and u_2.

We claim that we get a shock at time $t=t_s$ and position $x=x_s$:

Find the intersection of the characteristics starting at $x=0$ and $x=1$: Thus, solve

$$(1-2u_1)t = 1+(1-2u_2)t \Leftrightarrow t = \frac{1}{2(u_2-u_1)}$$, giving

$$x = (1-2u_1)\frac{1}{2(u_2-u_1)} = \underline{\frac{1-2u_1}{2(u_2-u_1)}}$$

For a characteristic starting at x_0 for $0<x_0<1$ we have

$$x\left(\frac{1}{2(u_2-u_1)}\right) = x_0 + \frac{1-2u_1}{u_2-u_1} - 2\frac{(u_2-u_1)x_0}{2(u_2-u_1)} = \underline{\frac{1-2u_1}{2(u_2-u_1)}}$$

Thus, all these characteristics pass through

$$(t_s, x_s) = \left(\frac{1}{2(u_2-u_1)}, \frac{1-2u_1}{2(u_2-u_1)}\right).$$

To find the position of the shock for $t>t_s$, $x=S(t)$, we have

$$S'(t) = \frac{J(u_+)-J(u_-)}{u_+-u_-} \quad \text{(Rankine-Hugniot)}$$

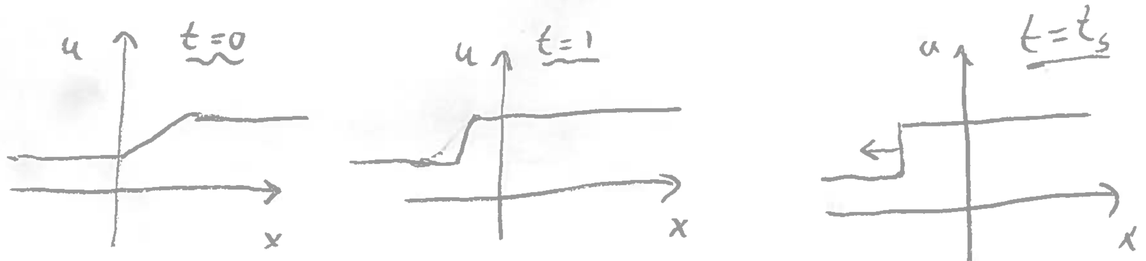
Here $J(u) = u(1-u)$, and $u_- = u_1$, $u_+ = u_2$. So

$$S' = \frac{u_2(1-u_2)-u_1(1-u_1)}{u_2-u_1} = 1-(u_1+u_2)$$

giving

$$S(t) = x_s + (t-t_s)(1-(u_1+u_2)), \quad t>t_s.$$

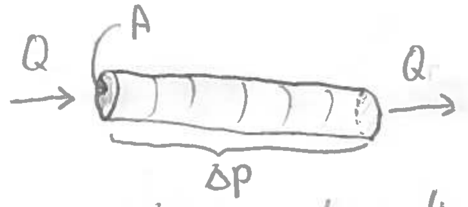
Thus, if $u_1+u_2>1$, the position of the shock moves backwards.



Solutions when $\frac{1}{2}<u_1<u_2$. Look at the characteristics for case c) in the previous figure.

2-phase flow in porous media, Buckley-Leverett equation

Single-phase flow



Consider a porous medium (e.g. a sandstone) shaped like a cylinder where the surface is sealed except at the two circular ends with area A. The pore space is filled with a liquid having viscosity μ. We will assume the cylinder, or plug, has small diameter, so that all flow is horizontal. Thus, we consider a one dimensional model for the flow in the plug.

If we apply a pressure drop over the core, fluid will come out downstream with a volumetric flux

$$(1) \qquad Q = -\left(\frac{k}{\mu}\frac{dp}{dx}\right)A \quad , \quad [Q] = m^3/s \, ,$$

where $-\frac{dp}{dx}$ is the (positive) pressure drop, and where k is the absolute permeability of the porous medium. $[k] = m^2$, and is a measure of the porous medium's ability to conduct a fluid through it's pore space.

$k = 10^{-12}\,m^2$ is considered a high (or good) permeability for a sandstone. (1) is usually stated for the volumetric flux density, q, and is called Darcy's law

$$(2) \qquad q = -\frac{k}{\mu}\frac{dp}{dx}$$

In the following, we assume k and μ are constant.

2-phase flow

When the pore space contains two immiscible fluids, say oil and water, the situation is more complicated. Firstly, buoyancy due to different mass density is an issue. And then there are capillary forces. We will disregard the direct inclusion of these mechanisms in our model.

When two phases (or fluids) are present, they both inhibit the others ability to flow in the porous medium due to capillary forces. This effect is modelled using two functions, the relative permeability to water, $k_{rw}(S_w)$, and the relative permeability to oil, $k_{ro}(S_w)$.

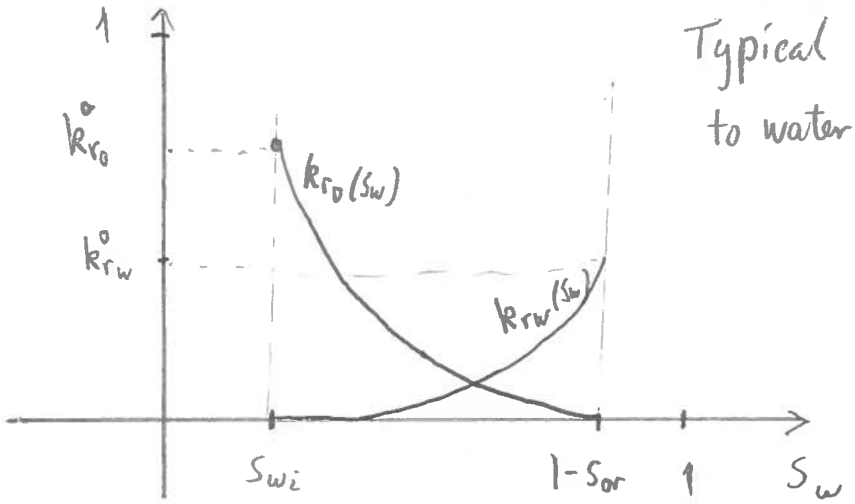
Here, S_w is the fraction of the pore space containing water, the water saturation. (Thus, the oil saturation S_o is $1-S_w$.

If q_w and q_o are the volumetric flux densities of water and oil, Darcy's law for two-phase flow is

$$q_w = -k_{rw}(S_w)\frac{k}{\mu_w}\frac{dp}{dx}, \quad (3a)$$

$$q_o = -k_{ro}(S_w)\frac{k}{\mu_o}\frac{dp}{dx}. \quad (3b)$$

We expect that q_w increases with increasing S_w, and that q_o decreases as S_w increases.



Typical relative permeability to water and oil.

S_{wi}, the <u>irreducible water saturation</u>, is the smallest water saturation where water still can flow, and S_{or}, the <u>residual oil saturation</u>, is likewise the smallest oil saturation when oil can flow. Typically, $S_{wi} \approx S_{or} \approx 0.2$.

A very common way to model relative permeabilities are

$$k_{rw}(S_w) = k_{rw}^{o}\left(\frac{S_w - S_{wi}}{1-S_{wi}-S_{or}}\right)^{\alpha_w}, \quad S_w \in [S_{wi}, 1-S_{or}]$$

$$k_{ro}(S_w) = k_{ro}^{o}\left(\frac{1-S_{or}-S_w}{1-S_{wi}-S_{or}}\right)^{\alpha_o}, \quad S_w \in [S_{wi}, 1-S_{or}]$$

Here k_{rw}^{o}, k_{ro}^{o} are <u>relative permeability endpoints</u> (see figure) and α_w, α_o so-called <u>Corey exponents</u>.

Typically $k_{rw}^{o} < k_{ro}^{o}$ and $\alpha_w, \alpha_o \approx 2$ or 3 (always ≥ 1).

$S_{wi}, S_{or}, k_{rw}^{o}, k_{ro}^{o}, \alpha_w, \alpha_o$ are generally measured (or estimated) in the lab.

Mass conservation

The mass of each fluid in a control volume from $x = a$ to $x = b$ is

$$\left(\int_a^b \rho_i S_i \phi \, dx\right) A \quad , \quad i = w, o ,$$

where ρ_i, $i = w, o$, is mass density. A cross-sectional area of plug.

The mass flux densities are

$$J_i = \rho_i q_i \quad , \quad i = w, o .$$

Thus mass conservation on integral form is

$$\frac{d}{dt}\left(\int_a^b \rho_i(t,x) S_i(t,x) \phi(t,x) \, dx\right) + \rho_i(t,b) q_i(t,b)$$

$$- \rho_i(t,a) q_i(t,a) = 0 \quad \text{for } i = w, o . \qquad (4)$$

Assumptions: We assume the fluids are incompressible, i.e. that ρ_w and ρ_o are constant, and that $\phi = \text{const}$.

Then (4) simplifies to $(i = w, o)$

$$\phi \frac{d}{dt} \int_a^b S_i(t,x) \, dx + q_i(t,b) - q_i(t,a) = 0 , \qquad (5)$$

Also, assume k and μ_i, $i = w, o$, are constant. Then (5) on differential form is

$$\varphi \frac{\partial S_i}{\partial t} + \frac{\partial q_i}{\partial x} = 0 \quad , i = w, o. \qquad (6)$$

These two equations is the system of equations for our two unknowns, $S_w(t,x)$ and $p(t,x)$ (recall $S_o = 1 - S_w$).

Now, we can actually eliminate p, and obtain a single equation for S_w.

If we add the two equations in (6) we obtain

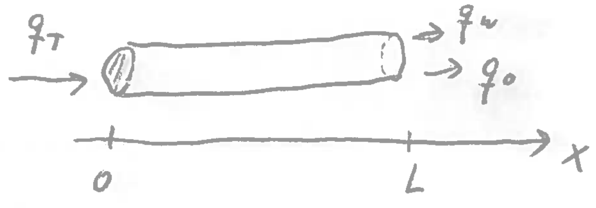
$$\varphi \frac{\partial}{\partial t}(S_w + S_o) + \frac{\partial}{\partial x}(q_w + q_o) = 0.$$

But $S_w + S_o = 1$, giving $\frac{\partial}{\partial x}(q_w + q_o) = 0$.

Let $q_T = q_w + q_o$ be the <u>total volumetric flux density</u>,

so $\frac{\partial q_T}{\partial x} = 0$, meaning that $q_T = q_T(t)$.

Note that $A q_T = Q_T$ is volume pushed into the plug at the inlet.



q_T is given by the boundary conditions at $x = 0$, and usually we inject one of the phases exclusively at $x = 0$.

In our example, we will inject water with constant flux density q_T.

Now (3a) and (3b) is written as

$$q_j = -\lambda_j(S_w)k\frac{dp}{dx}, \quad i = w, o,$$

where
$$\lambda_j(S_w) = \frac{k_{rj}(S_w)}{\mu_j}, \quad j = w, o,$$

$\lambda_j(S_w)$ are the <u>phase mobilities</u>, $j = w, o$.

$\lambda_j^o = \frac{k_{rj}^o}{\mu_j}$ are the <u>endpoint mobilities</u>, $j = w, o$.

Let us eliminate the pressure:

we have
$$q_T = q_w + q_o = -k\frac{dp}{dx}(\lambda_w(S_w) + \lambda_o(S_w))$$

$$\Leftrightarrow \quad \frac{dp}{dx} = -\frac{q_T}{k\,\lambda_T(S_w)}, \qquad (7)$$

where
$$\lambda_T(S_w) = \lambda_w(S_w) + \lambda_o(S_w)$$
is the <u>total mobility</u>.

Thus the water volumetric flux density is

$$q_w = -\lambda_w(S_w)k\frac{dp}{dx} = -\lambda_w(S_w)k\left(-\frac{q_T}{k\,\lambda_T(S_w)}\right)$$

$$\Leftrightarrow \quad q_w = \frac{\lambda_w(S_w)}{\lambda_T(S_w)}q_T = f_w(S_w)q_T,$$

where $f_w(S_w)$ is the water fractional flow curve.

Thus, equation (6) for $i = w$ now reads

$$\varphi \frac{\partial S_w}{\partial t} + q_T \frac{\partial f_w}{\partial x} = 0$$

$$\Leftrightarrow \boxed{\frac{\partial S_w}{\partial t} + \frac{q_T}{\varphi} f_w'(S_w) \cdot \frac{\partial S_w}{\partial x} = 0} \qquad (8)$$

(8) is the (relatively) famous Buckley-Leverett equation

So we have a hyperbolic conservation law with flux density

$$J = \frac{q_T}{\varphi} f_w(S_w)$$

For simplicity now, assume $S_{wi} = S_{or} = 0$, and that $\alpha_w = \alpha_o = 2$ ($S_{wi} = S_{or} = 0$ can always be done by scaling, but we skip that now). Also, set $S_w = S$.

Then

$$f_w(S) = \frac{\lambda_w^0 S^2}{\lambda_w^0 S^2 + \lambda_o^0 (1-S)^2} = \frac{S^2}{S^2 + M(1-S)^2},$$

where the dimensionless group

$$M = \frac{\lambda_o^0}{\lambda_w^0}$$

is the endpoint oil to water mobility ratio.

Scaling (8), we set L as length scale, $T = \frac{L\varphi}{q_T}$ as time scale, where L is the length of the plug. Thus, on final form

$$S_t + f_w'(S) S_x = 0 \qquad (9)$$

where $\quad f_w(S) = \dfrac{S^2}{S^2 + M(1-S)^2}$.

Then $f_w'(S) = \dfrac{2MS(1-S)}{(S^2+M(1-S)^2)^2}$, so we see that

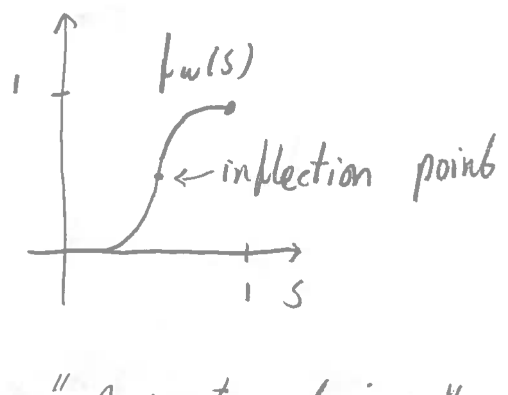
1) $f_w(0) = 0$, $f_w(1) = 1$

2) $f_w'(S) \geq 0$, so f_w is increasing.

3) $f_w'(0) = f_w'(1) = 0$, and by the mean value theorem $(f_w'(S))' = 0$ for some $S \in \langle 0,1 \rangle$.

Thus $f_w(S)$ has an inflection point. We conclude that $f_w'(S)$ has a maximum for an $S \in \langle 0,1 \rangle$.

(maximum since $f_w'(S) \geq 0$)



<u>Example</u> "<u>A water drive</u>"

To mimic oil recovery, or production, with two wells, one water injection well and one production well

we consider the following problem:

$$S_t + f_w'(S) S_x = 0 \ , \quad 0 \leq x \leq L \ , \quad t \geq 0 .$$

$S(0,x) = 0$ (initial condition)

$S(t,0) = 1$, $q_T > 0$ given. (boundary conditions)

The boundary condition $S(t,0) = 1$ is easily implemented by extending the domain towards the left, and state that initial conditions is

$$S(0,x) = \begin{cases} 1 & \text{for } x < 0 \\ 0 & \text{for } 0 < x < L \end{cases}$$

We know that characteristics are straight lines

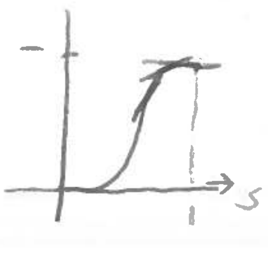
$$x(t) = (f_w'(S_0)(x_0))\, t + x_0 .$$

Since $f_w'(S) = 0$ for $S = 0$ and $S = 1$, all characteristics are simply $x = x_0$ (???), and nothing happens! However, for $q_T > 0$ there should be some action!

So, can the solution be expressed by a rarefaction wave?

I.e. $$\frac{x}{t} = f_w'(S) \qquad (9)$$

We know that $f_w'(1) = 0$, and that $f_w'(S)$ increases as S decreases, until we reach the inflection point of f_w, S_I.

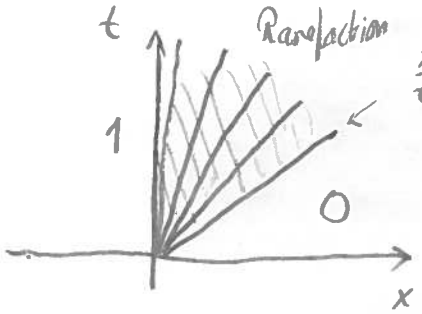


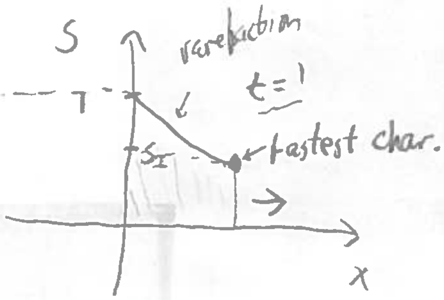
So for $S \in [S_I, 1]$, f_w' is monotonous, and we can solve (9) uniquely for S

when $S \in [S_F, 1]$.

So would $S = (f_w')^{-1}(\frac{x}{t})$ be a solution for

$$\frac{x}{t} < f_s'(S_I) \;?$$





In this case we would get a shock with $S_I = S_-$, $0 = S_+$

This is generally <u>not</u> the case,

Since the speed of the shock must match the speed of the fastest characteristic on the rarefaction wave.

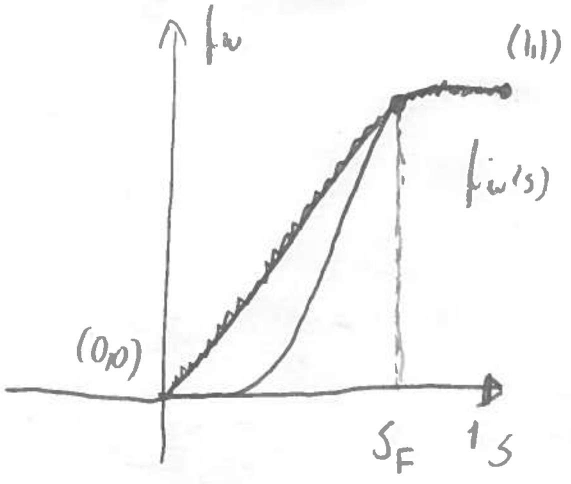
That is, we would in general <u>not</u> have

$$f_w'(S_I) = \frac{f_w(S_I) - f(0)}{S_I - 0}.$$

Instead we use (9) for $S \in [S_F, 1]$, where S_F solves

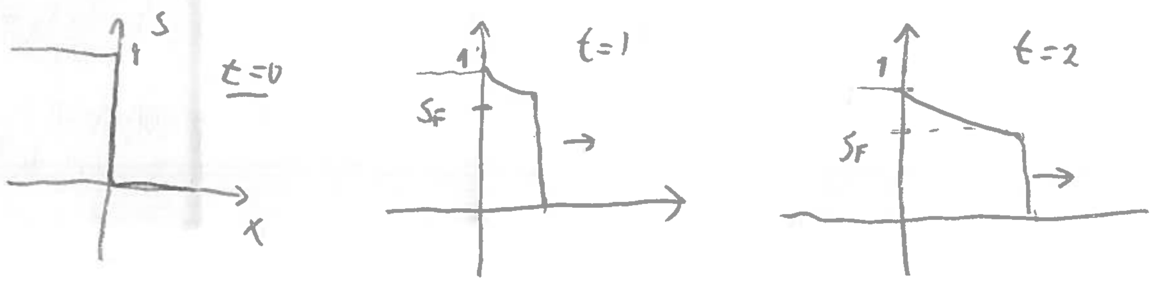
$$f_w'(S) = \frac{f_w(S) - f(0)}{S - 0}, \qquad (10)$$

so that the speed of the shock matches the speed of the fastest characteristic. S_F is is called the <u>front saturation</u>, and can be visulized using the "rubber band technique",



Imagine the graph of f_w is made of some solid material. Attach a rubber band at (1,1) above the graph. Attach the other end to (0,0) where the band is streched onto the graph of f_w. Then the point on the graph where the band leaves the curve gives S_F.

Thus, the solution looks like:



Going back to the example with

$$f_w(s) = \frac{s^2}{s^2+M(1-s)^2}$$

$$f_w'(s) = \frac{2MS(1-S)}{(s^2+M(1-s)^2)^2}$$

we find S_F by solving

$$f_w'(s) = \frac{f_w(s)}{s} \Leftrightarrow \frac{2MS(1-s)}{(s^2+M(1-s)^2)^2} = \frac{\frac{s^2}{s^2+M(1-s)^2}}{s}$$

$$\Leftrightarrow \quad \frac{2M(1-s)}{s^2+M(1-s)^2} = 1 \Leftrightarrow 2M - 2Ms = s^2 + M - 2Ms + s^2$$

$$\Leftrightarrow s^2 = \frac{M}{M+1} \Rightarrow \underline{S_F = \left(\frac{M}{M+1}\right)^{1/2}}.$$ We observe that $S_F \to 0$ as $M \to 0$, and $S_F \to 1$ as $M \to \infty$.

To find shock velocity U we know that

$$U = \frac{f_w(S_F)}{S_F} = \frac{S_F}{S_F^2 + M(1-S_F)^2} = \frac{1}{2[((M+1)M)^{1/2} - M]}$$

The (dimensionless) breakthrough time t_B signifies when the shock reaches the outlet at $x = 1$. Thus

$$1 = U t_B \Leftrightarrow \underline{t_B = 2[(M(M+1))^{1/2} - M]}.$$

Note, that for $t < t_B$ $q_w(t,1) = 0$, so only oil is produced.