Reynold's transport theorem

Let a velocity field $\vec{v}(t,\vec{x})$ be given in a subset $D \subset \mathbb{R}^3$. Let $R \subset D$, and for each $\vec{x}_0 \in R$, let $\vec{x}(t,\vec{x}_0)$ be the solution to

$$\frac{d\vec{x}}{dt} = \vec{v}(t,\vec{x}) \quad , \quad \vec{x}(0) = \vec{x}_0$$

Thus the curve $\vec{x}(t,\vec{x}_0)$ "flows along" the velocity field. If we let $R = R(0)$, we define

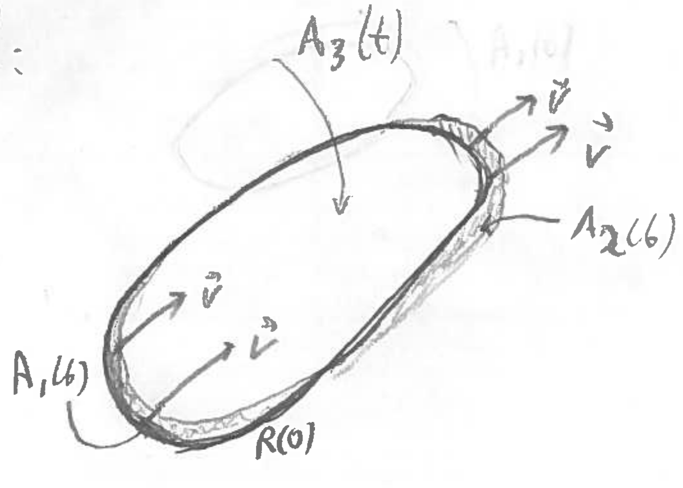
$$R(t) = \{ \vec{x}(t;\vec{x}_0) : \vec{x}_0 \in R(0) \} .$$

Reynolds transport theorem then states

$$\frac{d}{dt} \iiint_{R(t)} f(t,\vec{x})\, dV \Big|_{t=0} = \frac{d}{dt} \iiint_{R(0)} f(t,\vec{x})\, dV \Big|_{t=0} + \iint_{\partial R(0)} f(0,\vec{x})\, \vec{v}\cdot\vec{n}\, dS$$

where $\vec{n}$ points out of $R(0)$.

Proof:



Define:

$A_1(t) \equiv R(0) \sim R(t)$

$A_2(t) = R(t) \sim R(0)$

$A_3(t) = R(0) \cap R(t)$

Then

$R(0) = A_1(t) \cup A_3(t)$

$R(t) = A_2(t) \cup A_3(t)$

$A_1(t), A_2(t), A_3(t)$ are mutually disjoint. ②

Also define: $S_1 = \{ \vec{x} \in \partial R(0) \; : \; \vec{v}(0, \vec{x}) \cdot \vec{n} < 0 \}$

$$S_2 = \{ \vec{x} \in \partial R(0) \; : \; \vec{v}(0, \vec{x}) \cdot \vec{n} \geq 0 \}$$

So, $\partial R(0) = S_1 \cup S_2$.

Finally define $\Theta_j(t) = \iiint_{A_j(t)} f(t, \vec{x}) \, dV$, $j = 1, 2, 3$.

So $\iiint_{R(t)} f(t, \vec{x}) \, dV = \Theta_2(t) + \Theta_3(t)$, and

$$\iiint_{R(0)} f(t, \vec{x}) \, dV = \Theta_1(t) + \Theta_3(t)$$

We then have

$$\frac{d}{dt} \iiint_{R(t)} f(t, \vec{x}) \, dV \Big|_{t=0} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left(\iiint_{R(\Delta t)} f(t, \vec{x}) \, dV - \iiint_{R(0)} f(0, \vec{x}) \, dV \right)$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left(\Theta_2(\Delta t) + \Theta_3(\Delta t) - \Theta_3(0) \right)$$

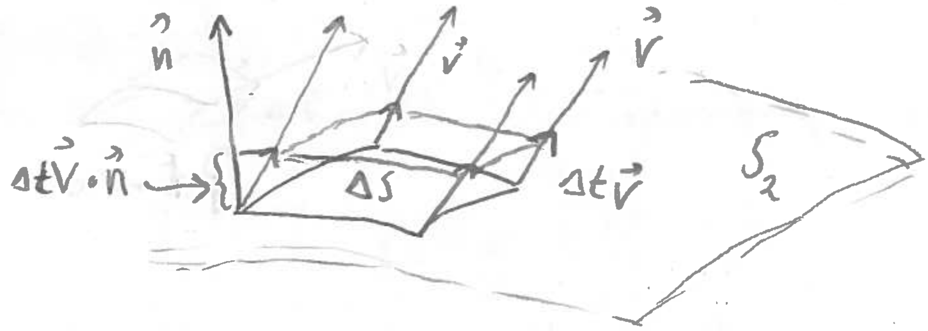
$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[\left(\Theta_3(\Delta t) + \Theta_1(\Delta t) - \Theta_3(0) \right) + \left(\Theta_2(\Delta t) - \Theta_1(\Delta t) \right) \right]$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[\iiint_{R(0)} f(t, \vec{x}) \, dV - \iiint_{R(0)} f(0, \vec{x}) \, dV \right]$$

$$+ \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[\Theta_2(\Delta t) - \Theta_1(\Delta t) \right] =$$

$$= \frac{d}{dt} \iiint_{R(0)} f(t, \vec{x})\, dv \Big|_{t=0} + \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left(\theta_2(\Delta t) - \theta_1(\Delta t) \right)$$

On S_2 we have $\vec{v} \cdot \vec{n} \geq 0$. During time $[0, \Delta t]$ a point on S_2 moves $\Delta t \vec{v}$ <u>out</u> from $R(0)$.



View $A_2(\Delta t)$ as a thin layer over S_2 with thickness $\Delta t \vec{v} \cdot \vec{n}$, where $\vec{n}$ is unit normal vector to S_2.

Thus, $\theta_2(\Delta t) = \iiint_{A_2(\Delta t)} f(\Delta t, \vec{x})\, dV \approx \iint_{S_2} f(\Delta t, \vec{x}) (\Delta t \vec{v}) \cdot \vec{n}\, dS$

$$= \Delta t \iint_{S_2} f(\Delta t, \vec{x})\, \vec{v} \cdot \vec{n}\, dS \Rightarrow$$

$$\lim_{\Delta t \to 0} \frac{\theta_2(\Delta t)}{\Delta t} = \iint_{S_2} f(0, \vec{x})\, \vec{v} \cdot \vec{n}\, dS.$$

On S_1, $\vec{v} \cdot \vec{n} < 0$ and $A_1(t)$ is a thin layer over S_1 with thickness $-\Delta t \vec{v} \cdot \vec{n}$. And as with $\theta_2(\Delta t)$ we have $\theta_1(\Delta t) \approx -\Delta t \iint_{S_1} f(\Delta t, \vec{x})\, \vec{v} \cdot \vec{n}\, dS$.

We have shown that

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(\Theta_2(\Delta t) - \Theta_1(\Delta t)\right) = \iint_{S_2} f(0,\vec{x})\,\vec{v}\cdot\vec{n}\,dS + \iint_{S_1} f(0,\vec{x})\,\vec{v}\cdot\vec{n}\,dS$$

$$= \iint_{\partial R(0)} f(0,\vec{x})\,\vec{v}\cdot\vec{n}\,dS,$$

and the proof is complete.

Navier-Stoke's equations, fluid mechanics

We consider flow of a fluid (liquid / or gas) in 3 spatial dimensions. At each time and position $(t,\vec{x})$ the fluid is assumed to have a well defined

mass density: $\rho = \rho(t,\vec{x})$

velocity : $\vec{v} = \vec{v}(t,\vec{x}) = [v_1(t,\vec{x}), v_2(t,\vec{x}), v_3(t,\vec{x})]$

energy : $e = e(t,\vec{x})$

We will obtain 5 equations for these 5 unknowns. The 5 equations express

1) mass conservation
2) energy conservation
3) 3 equations for momentum conservation.

Mass conservation

Let $R(t)$ be a domain following the flow. Then, since no mass is created or lost, we have

$$\frac{d}{dt}\iiint_{R(t)} \rho \, dV = 0 .$$

Using Reynolds transport theorem we obtain

$$\frac{d}{dt}\iiint_{R(0)} \rho \, dV + \iint_{\partial R(0)} \rho \vec{v}\cdot\vec{n} \, dS = 0 . \qquad (1)$$

Moving $\frac{d}{dt}$ inside the integral, and using the divergence theorem, we have

$$\iiint_{R(0)} \left(\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) \right) dV = 0$$

Since $R(0)$ is arbitrary (and $\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v})$ continuous) we have

$$\frac{\partial \rho}{\partial t} + \hat{\nabla}\cdot(\rho\vec{v}) = 0 . \qquad (2)$$

(2) is mass conservation on differential form, while (1) is this law on integral form.

Note that (2) is non-linear as it contains the product $\rho\vec{v}$ between unknowns.

Special case: Incompressible flow.

When the fluid is incompressible, the mass density is constant. So, $\frac{\partial \rho}{\partial t} = 0$ giving

$$\vec{\nabla}\cdot(\rho\vec{v}) = 0 \Leftrightarrow \rho\vec{\nabla}\cdot\vec{v} = 0 \Leftrightarrow \underline{\vec{\nabla}\cdot\vec{v} = 0}$$

Conclusion: For incompressible fluids we must have $\boxed{\vec{\nabla}\cdot\vec{v} = 0}$.

This is very reasonable as $\vec{\nabla}\cdot\vec{v} = 0 \Rightarrow 0 = \iiint_R \vec{\nabla}\cdot\vec{v}\, dV$

Divergence theorem $\Leftrightarrow 0 = \iint_{\partial R} \vec{v}\cdot\vec{n}\, dS$, i.e. the net flow out of R is zero. If it was not, mass would accumulate (or the opposite) and the density change.

Momentum conservation

Momentum conservation is really Newton's 2. law "force = mass. acceleration", that is,

$\vec{F} = \frac{d}{dt} m\vec{v}$ for a rigid body with mass m,

where $m\vec{v}$ is the (linear) momentum of the body.

We define the momentum density as

$$\rho\vec{v} \quad ,$$

and the momentum of a region $R(t)$ going with the flow as

$$\iiint_{R(t)} \rho\vec{v}\, dV \; .$$

Conservation of momentum is

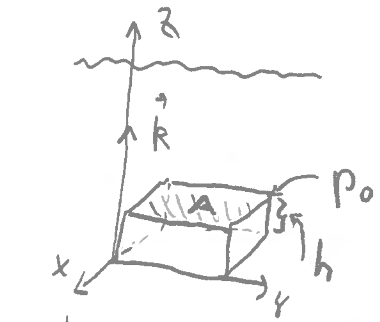
source for momentum

$$\frac{d}{dt}\iiint_{R(t)} \rho\vec{v}\, dV = \sum \vec{F} = \text{sum of all forces acting on } R(t)$$

or $$\frac{d}{dt}\iiint_{R(t)} \rho\vec{v}\, dV = \vec{F}^B_{R(t)} + \vec{F}^S_{R(t)}$$

where $\vec{F}^B$ is <u>body force</u>, and $\vec{F}^S$ <u>surface force</u>.
$\vec{F}^B$ is typically gravity forces, while $\vec{F}^S$ is the force the surrounding fluid acts with on the <u>surface</u> of $R(t)$.

Example



A box with density ρ_1 and volume Ah is submerged in a liquid with density ρ_2. What is the total force acting on the box?

Body force is gravity: $\vec{F}^B = -\rho_1 h A g \vec{k}$, g acceleration of gravity

Surface force from pressure:

Pressure acts on the surface of the box, with a force in the opposite direction of the outpointing unit normal $\vec{n}$.

By common reason the forces on the 4 vertical sides of the box cancel out.

On the top horizontal side, the force is

$$-p_0 A \vec{k}, \quad \text{(downwards)}$$

where p_0 is the pressure at this depth.

On the bottom side the hydrostatic pressure is

$$p_0 + \rho_2 g h, \text{ giving force}$$

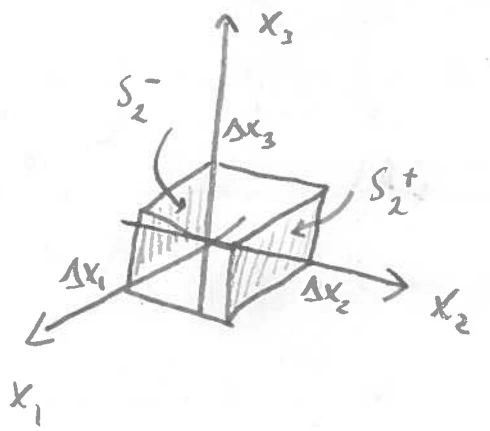
$$(p_0 + \rho_2 g h) A \vec{k} \quad \text{acting upwards.}$$

Thus
$$\vec{F}^s = A(p_0 + \rho_2 g h)\vec{k} - p_0 A \vec{k}$$
$$= (\rho_2 h A) g \vec{k} \quad \text{(Archimedes' law)}$$

Thus, the total force is
$$\vec{F} = \vec{F}^B + \vec{F}^s = (\rho_2 - \rho_1) g h .$$

Surface forces

The so-called stress tensor $\sigma = \sigma(t, \vec{x})$ is a 3×3-matrix, defined such that
$$(\sigma \vec{n}) \Delta S$$
is the surface force acting on a small surface ΔS with unit normal $\vec{n}$.



Thus, the force in the i-direction on surface S_2^+ is $\sigma_{i2}(x_1, \Delta x_2, x_3)\, \Delta x_1 \Delta x_3$

Since $\vec{n}$ has opposite direction on S_2^-, the force in the i-direction on S_2^- is $-\sigma_{i2}(x_1, 0, x_3)\, \Delta x_1 \Delta x_3 .$

So the net force in the i-direction on S_2^+ and S_2^- is

$$\left(\sigma_{i2}(x_1, \Delta x, x_3) - \sigma_{i2}(x_1, 0, x_3)\right)\Delta x_1 \Delta x_3$$

$$\approx \left(\frac{\partial \sigma_{i2}}{\partial x_2}(x_1, x_2^{\epsilon}, x_3)\Delta x_2\right)\Delta x_1 \Delta x_3 \quad \text{for } x_2^{\infty} \in [0, \Delta x_2]$$

$$= \frac{\partial \sigma_{i2}}{\partial x_2}(x_1, x_2^{\wedge}, x_3)\,\Delta V .$$

Consequently, the total surface force in the i-direction is

$$\left(\sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j}\right)\Delta V ,$$

Thus, the surface force in the i-direction on $R(t)$ is

$$\iiint_{R(t)} \left(\sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j}\right) dV$$

By defining $\vec{\nabla}\cdot\sigma$ as the vector with compont i:

$$(\vec{\nabla}\cdot\sigma)_i = \sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j} ,$$

we find $$\vec{F}^s = \iiint_{R(t)} \vec{\nabla}\cdot\sigma \, dV$$

Defining $\vec{f}^B$ as

$$\vec{F}^B = \iiint_{R(t)} \vec{f}^B \, dV \, ,$$

i.e. $\vec{f}^B$ is the body force density, we obtain

$$\frac{d}{dt} \iiint_{R(t)} \rho \vec{v} \, dV = \iiint_{R(t)} (\vec{f}^B + \vec{\nabla} \cdot \sigma) \, dV \qquad (6)$$

Cauchy stress tensor

Let $\sigma_{ij} = -\delta_{ij} p + \tilde{\sigma}_{ij}$,

where $\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$ is the Kroneker delta symbol.

The $-\delta_{ij} p$ part, p pressure, acts always normal to the surface, while the $\underline{\tilde{\sigma}_{ij}}$ part represent the <u>friction forces</u>.

How do we define $\tilde{\sigma}_{ij}$? If $\tilde{\sigma}$ is to represent frictional effects, it should depend on the variation in space of the field variables $\rho, \vec{v}$, and e.

, i.e. depend on the various derivatives

$$\frac{\partial \rho}{\partial x_j}, \frac{\partial e}{\partial x_j}, \frac{\partial v_i}{\partial x_j}, \quad i,j = 1,2,3.$$

Now, this is a fact: In order for our equations to be valid in all inertial systems (Galilean invariance) $\tilde{\sigma}$ cannot depend on $\frac{\partial \rho}{\partial x_j}, \frac{\partial e}{\partial x_j}$.

To our level of approximation, we assume $\tilde{\sigma}$ is linear in $\frac{\partial v_i}{\partial x_j}$, $i,j = 1..3$, i.e.

$$\tilde{\sigma}_{ij} = \sum_{m,n} \alpha_{ijmn} \frac{\partial v_m}{\partial x_n}, \quad \alpha_{ijmn} \text{ constants.}$$

Further physical restrictions (we have an isotropic fluid) narrow $\tilde{\sigma}$ down to

$$\tilde{\sigma}_{ij} = a\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) + b\,\delta_{ij}\,\vec{\nabla}\cdot\vec{v},$$

for some constants a, b.

We want to split the matrix $\tilde{\sigma}$ in two terms such that the first term has trace zero.

We see that

$$\mathrm{trace}\left\{a\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)\right\}_{i,j}$$

$$=\sum_{i=1}^{3} a\left(\frac{\partial v_i}{\partial x_i}+\frac{\partial v_i}{\partial x_i}\right) = 2a\,\vec{\nabla}\cdot\vec{v}$$

$$\mathrm{trace}\left\{b\,\delta_{ij}\,\vec{\nabla}\cdot\vec{v}\right\} = 3b\,\vec{\nabla}\cdot\vec{v}$$

Thus writing,

$$\hat{\sigma}_{ij} = a\underbrace{\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}-\frac{2}{3}\delta_{ij}\,\vec{\nabla}\cdot\vec{v}\right)}_{\text{traceless}} + \left(b+\tfrac{2}{3}a\right)\delta_{ij}\,\vec{\nabla}\cdot\vec{v}$$

we obtain the final expression for $\hat{\sigma}_{ij}$:

$$\boxed{\hat{\sigma}_{ij} = \eta\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}-\frac{2}{3}\delta_{ij}\,\vec{\nabla}\cdot(\vec{v})\right)+\zeta\,\delta_{ij}\,\vec{\nabla}\cdot\vec{v}}$$

Here $\eta = a$, $\zeta = b+\frac{2}{3}a$ are the shear viscosity and the bulk (or volume) viscosity, respectively.

The shear viscosity η is the "standard" viscosity of the fluid. While the bulk visosity ζ is often disregarded, but can be important for rapidly expanding gases, and ζ plays a role in damping of sound waves.

Recall the conservation equation for momentum (6):

$$\frac{d}{dt}\iiint_{R(t)} \rho\vec{v}\,dV = \iiint_{R(t)} \vec{f}^{B}\,dV + \iiint_{R(t)} \vec{\nabla}\cdot\sigma\,dV$$

Thus, we need to calculate $\hat{\nabla}\cdot\sigma$:

We have $(\vec{\nabla}\cdot\sigma)_i = \sum_{j=1}^{3} \frac{d\sigma_{ij}}{dx_j}$ and

$$\sigma_{ij} = -\delta_{ij}p + \eta\left(\frac{dv_i}{dx_j} + \frac{dv_j}{dx_i} - \frac{2}{3}\delta_{ij}\vec{\nabla}\cdot\vec{v}\right) + \xi\delta_{ij}\vec{\nabla}\cdot\vec{v}$$

$$\text{Thus, } (\nabla\cdot\sigma)_i = -\frac{dp}{dx_i} + \eta\left[\sum_{j=1}^{3}\left(\frac{d^2v_i}{dx_j^2} + \frac{d^2v_j}{dx_i dx_j}\right) - \frac{2}{3}\frac{d}{dx_i}(\vec{\nabla}\cdot\vec{v})\right]$$

$$+ \xi\frac{d}{dx_i}(\vec{\nabla}\cdot\vec{v})$$

$$= -\frac{dp}{dx_i} + \eta\left[\nabla^2 v_i + \frac{d}{dx_i}\left(\vec{\nabla}\cdot\vec{v} - \frac{2}{3}\vec{\nabla}\cdot\vec{v}\right)\right]$$

$$+ \xi\frac{d}{dx_i}(\vec{\nabla}\cdot\vec{v}).$$

So, we see

$$\boxed{\vec{\nabla}\cdot\sigma = -\vec{\nabla}p + \eta\left[\nabla^2\vec{v} + \frac{1}{3}\vec{\nabla}(\vec{\nabla}\cdot\vec{v})\right] + \xi\vec{\nabla}(\vec{\nabla}\cdot\vec{v})}.$$

Thus, the right hand side of (6) is:

$$\iiint_{R(t)} \left(\vec{f}^{\,B} - \vec{\nabla}p + \eta\left[\nabla^2\vec{v} + \frac{1}{3}\vec{\nabla}(\vec{\nabla}\cdot\vec{v})\right] + \xi\vec{\nabla}(\vec{\nabla}\cdot\vec{v})\right) dV \quad (7)$$

For the left hand side of (6) we use Reynolds transport theorem:

$$\frac{d}{dt}\iiint_{R(t)} \rho v_i \, dV = \iiint_{R(t)} \frac{d}{dt}(\rho v_i)\, dV + \iint_{\partial R(t)} \rho v_i \, \vec{v}\cdot\vec{n}\, dS$$

$i = 1, 2, 3,$

$$= \iiint_{R(t)} \left(\frac{d}{dt}(\rho v_i) + \vec{\nabla}(\rho_i\, \rho\vec{v}) \right) dV$$

$$= \iiint_{R(t)} \left(\frac{d\rho}{dt} v_i + \rho \frac{dv_i}{dt} + \rho_i\, \vec{\nabla}.(\rho\vec{v}) + \rho\vec{v}.(\vec{\nabla} v_i) \right) dV$$

Since $\vec{\nabla}(\rho\vec{v}) = -\frac{d\rho}{dt}$ from mass conservation, we obtain

$\iiint_{R(t)} \rho \left(\frac{dv_i}{dt} + \vec{v}.(\vec{\nabla} v_i) \right) dV$, or on vectorform

$\iiint_{R(t)} \rho \left(\frac{d\vec{v}}{dt} + (\vec{v}.\vec{\nabla})\vec{v} \right) dV$. Thus, from (7) we

arrive at the final equation:

$$\boxed{\rho\left(\frac{d\vec{v}}{dt} + (\vec{v}.\vec{\nabla})\vec{v}\right) = -\vec{\nabla}p + \eta\left(\nabla^2\vec{v} + \frac{1}{3}\vec{\nabla}(\vec{\nabla}.\vec{v})\right) + \zeta\,\vec{\nabla}(\vec{\nabla}.\vec{v}) + \vec{f}^B} \quad (9)$$

Special cases: Incompressible fluid, $\rho = \text{const.}$

$$\rho\left(\frac{d\vec{v}}{dt} + (\vec{v}.\vec{\nabla})\vec{v}\right) = -\vec{\nabla}p + \eta\nabla^2\vec{v} + \vec{f}^B$$

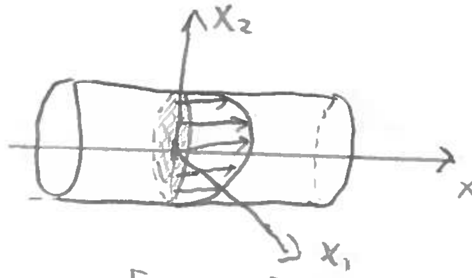
$$\Leftrightarrow \quad \frac{d\vec{v}}{dt} + (\vec{v}.\vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla}p + \left(\frac{\eta}{\rho}\right)\nabla^2\vec{v} + \frac{1}{\rho}\vec{f}^B$$

Inviscid fluid $\eta = \zeta = 0$

$$\rho\left(\frac{d\vec{v}}{dt} + (\vec{v}.\vec{\nabla})\vec{v}\right) = -\vec{\nabla}p + \vec{f}^B$$

Example: Laminar flow in a circular tube.

Assume steady state laminar flow (low Reynolds number) in a circular tube of radius R, aligned in direction x_3.



All flow is in the x_3-direction, so

$$\vec{v} = \begin{bmatrix} 0 \\ 0 \\ v_3(r) \end{bmatrix}, \text{ where } r \text{ is the polar coordinate}$$

for radius in the $x_1 x_2$-plane. Since $\frac{dv_3}{dt} = 0$ (steady state), and $\vec{v} \cdot \vec{\nabla} = v_3 \frac{d}{dx_3}$, we have

$$(\vec{v} \cdot \vec{\nabla}) \vec{v} = v_3 \frac{d}{dx_3} \begin{bmatrix} 0 \\ 0 \\ v_3(r) \end{bmatrix} = \vec{0}.$$

Assume incompressibility, and from (9) we have

$$\vec{0} = -\vec{\nabla} p + \eta \nabla^2 \vec{v}.$$

Since we have steady state flow in x_3 direction, we have

$$\vec{\nabla} p = \begin{bmatrix} 0 \\ 0 \\ p_{x_3} \end{bmatrix}, \text{ where } p_{x_3} = \frac{dp}{dx_3} = \text{constant}.$$

$$\eta \nabla^2 \vec{v} = \eta \nabla^2 \begin{bmatrix} 0 \\ 0 \\ v_3(r) \end{bmatrix} = \eta \begin{bmatrix} 0 \\ 0 \\ \nabla^2 v_3(r) \end{bmatrix}$$

$\nabla^2 f(r) = f'' + \frac{1}{r} f'$ for polar coordinates

$$\eta \begin{bmatrix} 0 \\ 0 \\ \frac{d^2 v_3}{dx_1^2} + \frac{d^2 v_3}{dx_2^2} \end{bmatrix} = \eta \begin{bmatrix} 0 \\ 0 \\ v_3''(r) + \frac{1}{r} v_3'(r) \end{bmatrix}$$

This gives

$$p_{x_3} = \eta\left(v_3'' + \frac{1}{r} v_3'\right)$$

$$= \eta\left(\frac{1}{r}\left(r v_3'\right)\right)'$$

Thus $r v_3' = \frac{1}{2\eta} r^2 p_{x_3} + C$.

We see that $C = 0$ as $r \to 0$. This gives

$$v_3' = \frac{1}{2} r \, p_{x_3}/\eta$$

$$\Rightarrow v_3 = \frac{1}{4} \frac{r^2}{\eta} p_{x_3} + C_1$$

Now, the so-called <u>no-slip condition</u> says that the velocity at the wall of the tube is zero. This gives

$$v_3(R) = 0 \iff C = -\frac{1}{4} \frac{R^2}{\eta} p_{x_3}$$

Thus, the velocity is

$$\underline{v_3(r) = \frac{p_{x_3}}{4\eta}\left(R^2 - r^2\right)}$$

We see the velocity is inversely propotional to the shear viscosity, which makes sense.

The volumetric flux through the tube is (D_R is cross-section of tube)

$$Q = \iint_{D_R} v_3(r)\, dA = \int_0^{2\pi}\int_0^R \left(\frac{p_x}{4\eta}\left(R^2 - r^2\right)\right) r\,dr\,d\theta$$

$$= 2\pi \frac{p_x}{4\eta} \left[\left(\frac{R^2 r^2}{2} - \frac{1}{4} r^4 \right) \right]_0^R$$

$$= \underline{\frac{\pi}{8} \frac{p_x}{\eta} R^4}$$. Thus, we can measure η by applying a pressure drop over the tube, and measure the total flow rate Q, and find

$$\underline{\eta = \frac{\pi}{8} \frac{p_x}{Q} R^4}$$

The volumetric (or total) flow-rate allso gives us the average velocity

$$\bar{v} = \frac{Q}{A} = \frac{Q}{\pi R^2} = \frac{R^2}{8\eta} p_{x_3}$$

Comparing to Darcy's law

$$\bar{v} = \frac{k}{\eta} p_{x_3} \quad \text{(ignoring sign)}$$

we find the permeability of a circular tube is

$$\underline{k = R^2/8} \quad \text{(Pouseille's law)}$$

Thus, a tube with $R = 10^{-5}$ m has a permeability of

$$k = \frac{10^{-10}}{8} m^2 \approx 1.25 \cdot 10^{-11} m^2 = 12.5 \cdot 10^{-12} m^2 \approx 12.5 \text{ Darcy}$$

which is high permeability compared to the permeability of a sand stone.

Conservation of energy

We have "energy = force · distance", or in vectorform

$$dE = \vec{F}\cdot d\vec{r} = \vec{F}\cdot\vec{v}\,dt$$

For $R(t)$ flowing with the velocity, the energy transferred to $R(t)$ by the body- and surface forces during a short time Δt is

surface force density

$$\Big(\iiint_{R(t)} \vec{f}^{\,B}\cdot\vec{v}\,dV + \iint_{\partial R(t)} \vec{v}\cdot(\sigma\vec{n})\,dS\Big)\Delta t$$

Thus, energy conservation reads (e energy density)

$$\frac{d}{dt}\iiint_{R(t)} e\,dV = \iiint_{R(t)} \vec{f}^{\,B}\cdot\vec{v}\,dV + \iint_{\partial R(t)} \vec{v}\cdot(\sigma\vec{n})\,dS$$

$$\Leftrightarrow \iiint_{R(t)}\Big(\frac{\partial e}{\partial t} + \vec{\nabla}\cdot(e\vec{v})\Big) = \iiint_{R(t)}\Big(\vec{f}^{\,B}\cdot\vec{v} + \vec{\nabla}\cdot(\sigma\vec{v})\Big)dV,$$

giving
$$\boxed{\frac{\partial e}{\partial t} + \vec{\nabla}\cdot(e\vec{v}) = \vec{f}^{\,B}\cdot\vec{v} + \vec{\nabla}\cdot(\sigma\vec{v})} \qquad (10)$$

We can also express (10) in terms of the spesific thermal energy density $\mathcal{E}$: We have

$$e = \tfrac{1}{2}\rho|\vec{v}|^2 + \rho\mathcal{E}.$$

Insert this into (10) and express (10) through $\mathcal{E}$.