# Dynamic system, equilibrium points

## Dynamic system

$$\begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix} = \begin{bmatrix} f_1\left(x_1, x_2, \ldots, x_n\right) \\ f_2\left(x_1, x_2, \ldots, x_n\right) \\ \vdots \\ f_n\left(x_1, x_2, \ldots, x_n\right) \end{bmatrix} \quad \Leftrightarrow \quad \vec{x}' = \vec{f}\left(\vec{x}\right), \quad x\left(t\right) \in \sim^{n}$$

## Equilibrium points:

Solutions $\vec{x} = \vec{x}_e$ to $\vec{f}\left(\vec{x}\right) = \vec{0}$

# Stability

$\vec{x}_e$ is **stable** if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that if

$\left\| \vec{x}(0) - \vec{x}_e \right\| < \delta$, then $\left\| \vec{x}(t) - \vec{x}_e \right\| < \varepsilon$ for all $t > 0$.

$\vec{x}_e$ is **asymptotically stable** if $\vec{x}_e$ is stable, and there exists a $\delta > 0$

such that $\left\| \vec{x}(0) - \vec{x}_e \right\| < \delta$ implies $\left\| \vec{x}(t) - \vec{x}_e \right\| \to 0$ as $t \to \infty$.

$\vec{x}_e$ is **unstable** if it is not stable.

# Linearization

Assume $\vec{f}$ is twice continuously differentiable. Then

$$\vec{f}\left(\vec{x}+\vec{x}_0\right)=\vec{f}\left(\vec{x}_0\right)+d\vec{f}\left(\vec{x}_0\right)\vec{x}+O\left(\left\|\vec{x}\right\|^2\right),$$

where
$$\left(d\vec{f}\right)_{i,j}=\frac{\partial f_i}{\partial x_j},\quad i,j=1,2,..,n$$

is the Jacobi matrix. The linearization of $\vec{x}'=\vec{f}\left(\vec{x}\right)$ around an equlibrium point $\vec{x}_e$ is the linear equation

$$\vec{\tilde{x}}'=d\vec{f}\left(\vec{x}_e\right)\vec{\tilde{x}}$$

where

$$\vec{x}\left(t\right)=\vec{\tilde{x}}\left(t\right)+\vec{x}_e\,.$$

# Linear stability

Assume the nxn-matrix $A$ is nonsingular and diagonalizable. Then the only equilibrium point of the dynamical system

$$\vec{x}' = A\vec{x}$$

is $\vec{x}_e = \vec{0}$. The general solution is $\vec{x}(t) = C_1\vec{\chi}_1 e^{\lambda_1 t} + C_2\vec{\chi}_2 e^{\lambda_2 t} + \ldots. + C_n\vec{\chi}_n e^{\lambda_n t}$ ,

where $C_1,..,C_n$ are constants, $\vec{\chi}_1,..,\vec{\chi}_n$ linearly indenpendent eigenvectors, and $\lambda_1,..,\lambda_n$ corresponding eigenvalues. Therefore, $\mathrm{Re}\,\lambda_j,\ j = 1, 2, \ldots, n$, determine the stability properties of the equilibrium point $\vec{x}_e = \vec{0}$ .

# $2\times 2-$ matrices and eigenvalues

Let $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix},\ a,b,c,d \in \sim$ have eigenvalues $\lambda_1,\lambda_2$.

Then

$$\det(A)=ad-bc=\lambda_1\lambda_2$$
$$\operatorname{trace}(A)=a+d=\lambda_1+\lambda_2$$

So, if the eigenvalues have nonvanishing imaginary part, i.e , $\lambda_{1,2}=a\pm ib,\ b\neq 0,$ , then

$\operatorname{trace}(A)=2a$ . For real eigenvalues we see that $\det(A)>0$ if and only if they have

the same sign. In this case, $\operatorname{trace}(A)$ determines the common sign of the two eigenvalues.