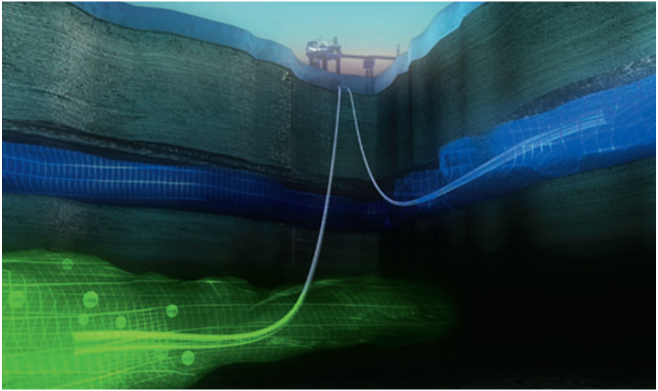
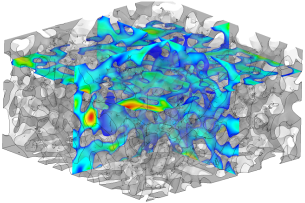
Mathematical Modeling Project fall 2019

Underground CO2 Storage



Multiphase flow in porous media

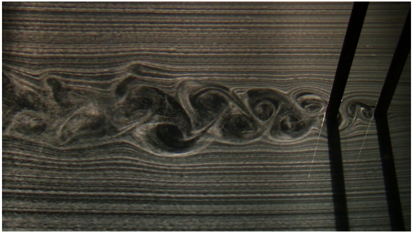
- The rock is porous
- We displace a water phase by a CO_2 phase.
- Multiphase flow in porous media



Flow equation Quiz

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0$$

$$\frac{\partial \rho u}{\partial t} + \nabla \cdot (\rho u \otimes u) = -\nabla p + \mu \Delta u + \frac{\mu}{3} \nabla(\nabla \cdot u) + f$$



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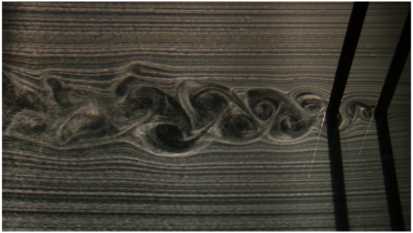
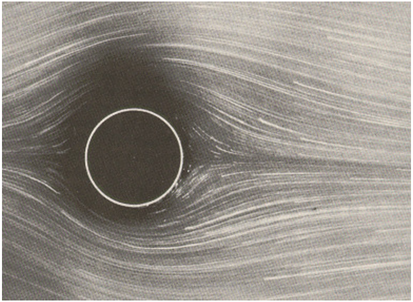


Navier-Stokes equation

Flow equation Quiz

$$\nabla \cdot u = 0$$
$$\mu \Delta u - \nabla p + f = 0$$



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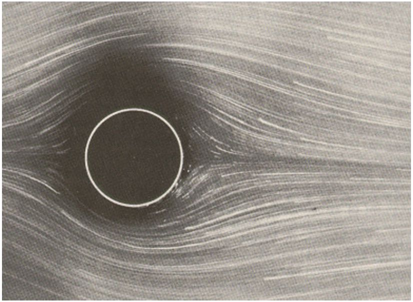

Incompressible Stokes equation

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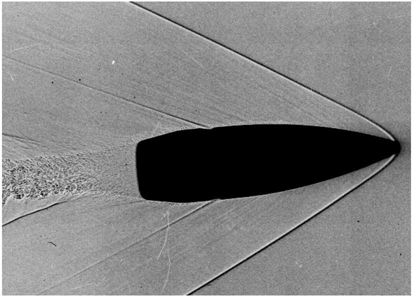

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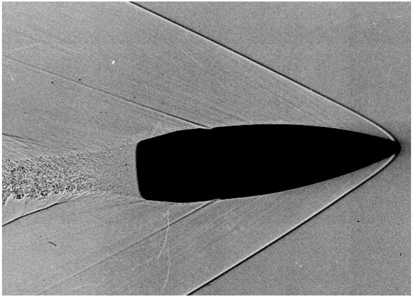
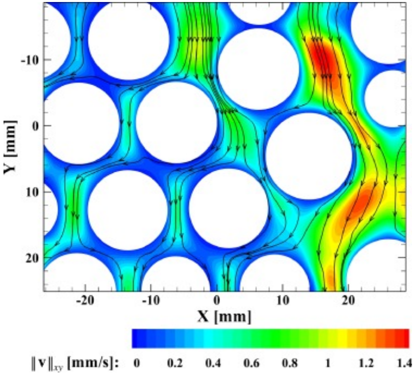


Euler equation

Darcy approximation

- Strong variation of the velocity at the pore scale



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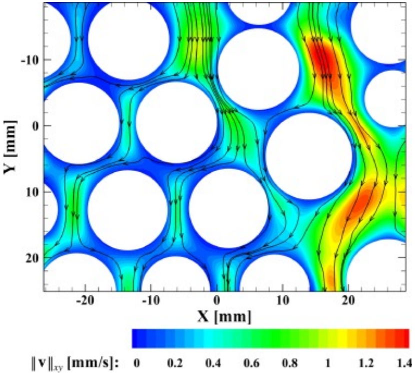
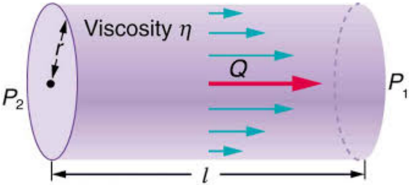


- We cannot approximate these oscillations. We use a velocity average or **Darcy velocity**.

Darcy approximation

- Poisefeuille flow



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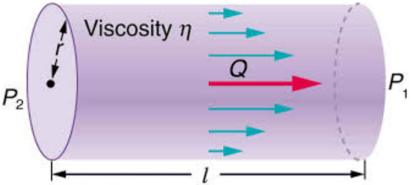

- We can compute the exact solution of the Stokes equation and obtain

$$\frac{Q}{\Delta P} = \frac{\pi R}{8\mu L}.$$

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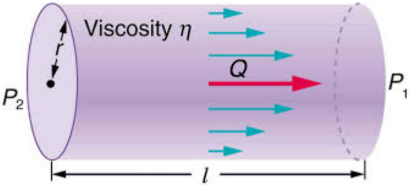


- We can compute the exact solution of the Stokes equation and obtain

$$\frac{Q}{\Delta P} = \frac{\pi R}{8\mu L}.$$

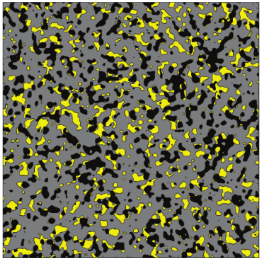
- By **homogenization**, we can show that there exist a permeability tensor $\boldsymbol{K}$ such that

$$u = -\frac{1}{\mu}\boldsymbol{K}\nabla p$$

for the Darcy's velocity u.

Immiscible flow

- Immiscible phases



- black : solid part
- gray : wetting phase
- yellow : non-wetting phase

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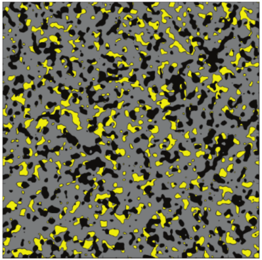

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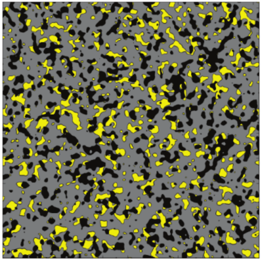

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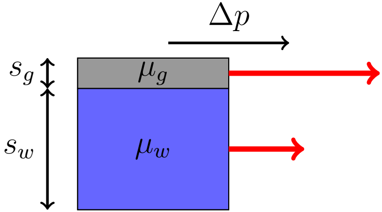
- Again, we cannot track all these details.
- We use upscaled variable:
  - **porosity** : ϕ
  - **saturation** : s_g and s_w, volume fraction of the pore volume occupied by the phase

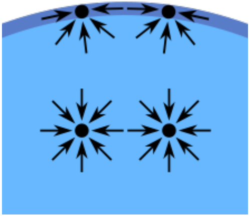
The effect of viscosity

- The Darcy's velocity is inversely proportional to the viscosity



Surface tension and Young Laplace equation

- Surface tension



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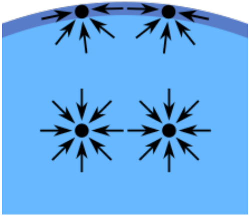
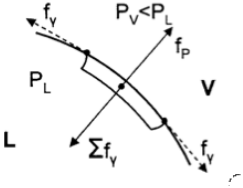


- An interface can withstand a pressure

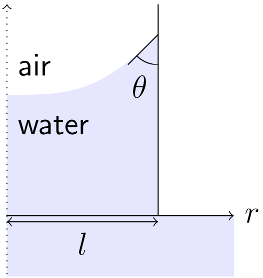


Young-Laplace equation

$$\Delta p = \gamma\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

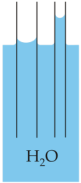
Shape of a meniscus

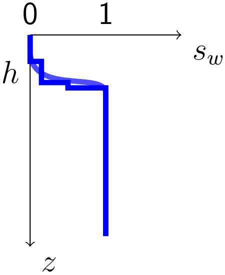
- The shape of a meniscus is the result of force balance between
  - Gravity forces
  - Surface tension between air and water
  - Affinity between water/air and container.



Capillary effects - homogenization

- We cannot model the details. We proceed with **homogenization**





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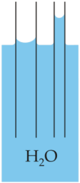
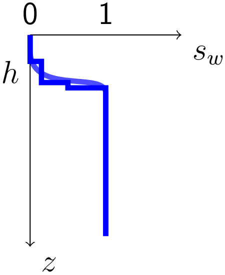
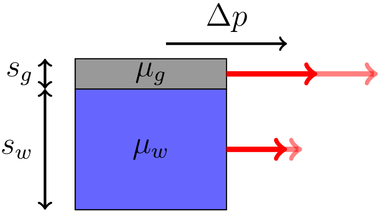




- The upscaled effects are given by the **relative permeability** and the **capillary pressure function**.

relative permeability

- The relative permeability accounts for the difficulty of a phase to flow when it is surrounded by another phase.



In this case, it is **relatively harder** for the gas to flow in because there is a lot of water.

Capillary pressure function

- One pressure per phase, p_g and p_w
- The capillary function is a function of saturation which relates the two pressure phases,

$$p_c(s) = p_w - p_g$$

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$$p_c(s) = p_w - p_g$$

- Consider p_w as reference pressure. Then, the gas pressure is

$$p_g = p_w - p_c(s).$$

- Property: p_c is **monotone decreasing**.

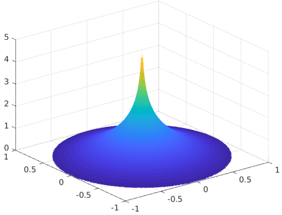
project description

Single phase flow

- Governing equation

$$\frac{\partial \rho\phi}{\partial t} + \nabla \cdot (-\frac{\rho}{\mu}\boldsymbol{K}\nabla p) = 0.$$

- **Q1,2**: Derive the equation and compute the solution for a linear or point injection well with radially symmetric boundary conditions.



Two-phase flow

- Governing equation

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$$\boldsymbol{u}_\alpha = -\frac{s_\alpha}{\mu_\alpha} \boldsymbol{K} \nabla p.$$

Q3: Derive the equation

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- **Fractional flow formulation**. In 1D and incompressible case, the equations can be written in the fractional flow formulation,

$$(\phi s)_t + (f(s)u)_x = 0.$$

Q4,5: Explain this derivation.

Hyperbolic conservation laws

- General form of an hyperbolic conservation law in 1D

$$s_t + f(s)_x = 0$$

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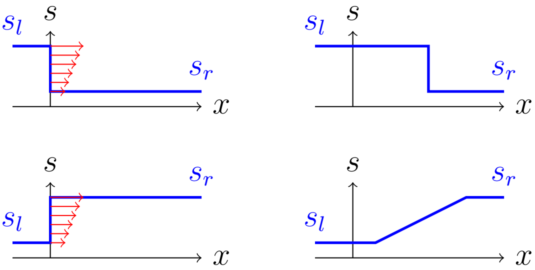
- Fundamental solutions for hyperbolic conservation laws are given by the **Riemann** problems
- Example: Burgers' equation: $s_t + (s^2)_x = 0$,



shock

rarefaction wave

Fractional flow formulation

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Q5: Compute the Riemann solution.

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- We add capillary pressure, $p_c(s)$.

$$(\phi s)_t + (f(s)u)_x - (g(s)s_x)_x = 0$$

Q6,7: Derive the equation. Find expressions for f and g. Solve the equation numerically.

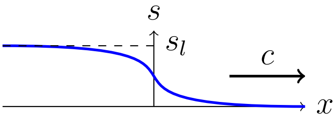
Traveling wave solution

- The equation

$$(\phi s)_t + (f(s)u)_x - (g(s)s_x)_x = 0$$

is no longer a hyperbolic conservation law. We do not have Riemann solution.
Q8: Investigate the existence of traveling waves,

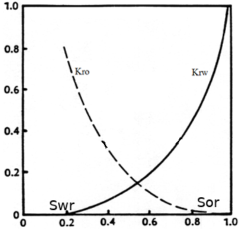
$$s(t,x) = \hat{s}(x - ct)$$



Find s_l and velocity c.

Relative permeability

- Relative permeabilities are empirical data



- We have

$$u_\alpha = \frac{k_{r\alpha}(s_\alpha)}{\mu_\alpha} K \nabla p$$

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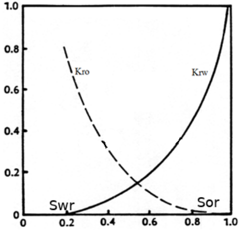

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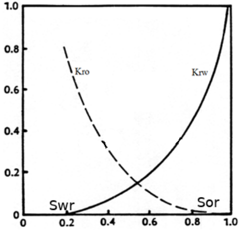

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- **Q9,10**: What are the Riemann solutions and the traveling waves for the corresponding fractional flow formulation?

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- The effects of gravity have the smallest timescale.

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- **Q11,12**: Explain how the reduced model is obtained. Consider the compressible case.

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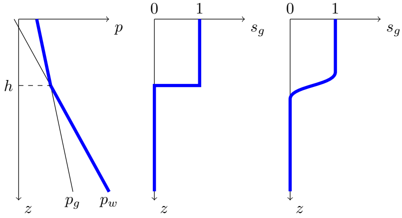
- The phases segregate so that the gas remains at the top of the water
- The vertical equilibrium equations (without capillary pressure) are

$$\lambda_\alpha(s_\alpha)\frac{k}{\mu_\alpha}(\frac{\partial p}{\partial z} - \rho_\alpha g) = 0,$$

- **Q13,14**: Compute the solution of the vertical equilibrium with and without capillary pressure



Vertical Equilibrium Approximation - Multi-Phase

- Governing equation for the **reduced** problem:

$$\frac{\partial \phi S_\alpha}{\partial t} + \nabla_{\parallel} \cdot \boldsymbol{u}_\alpha = 0,$$
$$\boldsymbol{u}_\alpha = -\Lambda_\alpha(S_\alpha) k \nabla_{\parallel} P_\alpha,$$
$$P_c(S_g) = P_w - P_g.$$

Vertical Equilibrium Approximation - Multi-Phase

- Governing equation for the **reduced** problem:

$$
\begin{aligned}
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- **Q15**: Derive this model equation

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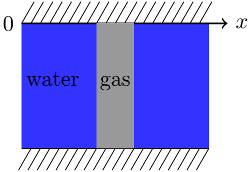
- **Q15**: Derive this model equation
- Remarkably, the model equation takes the same form as the original 2D problem with relative permabilities and capillary pressure.

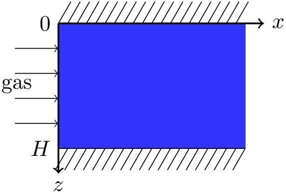
Vertical Equilibrium Approximation - Multi-Phase

- Fractional flow formulation

$$\frac{\partial \phi s}{\partial t} + \frac{\partial}{\partial x}(f(s)u - g(s)s_x) = 0.$$

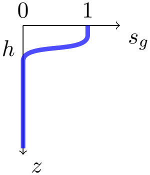
- **Q16,17**: Derive fractional flow formulation. Implement numerical scheme. Solve for the following inputs

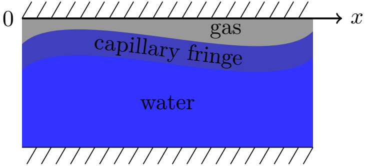




Vertical Equilibrium Approximation - Capillary fringe

- We add capillary effects and obtain a capillary fringe

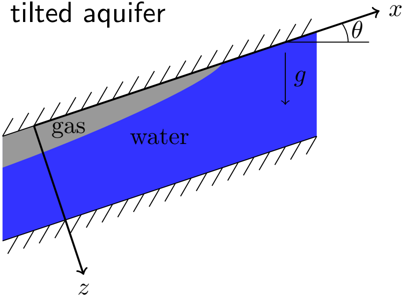




- **Q18,19**: Find a relation between S_g and h (where $p_c(s_g(h)) = 0$). Derive the reduced model.

Tilted aquifer

- We consider a tilted aquifer



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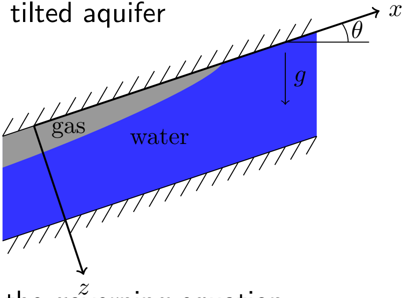

- **Q20**: Derive the governing equation

$$\frac{\partial \phi s_\alpha}{\partial t} - \frac{\partial}{\partial x}(\lambda_\alpha(s_\alpha)k(\frac{\partial p}{\partial x} + \rho_\alpha g \sin\theta)) - \frac{\partial}{\partial z}(\lambda_\alpha(s_\alpha)k(\frac{\partial p}{\partial z} - \rho_\alpha g \sin\theta)) = 0$$

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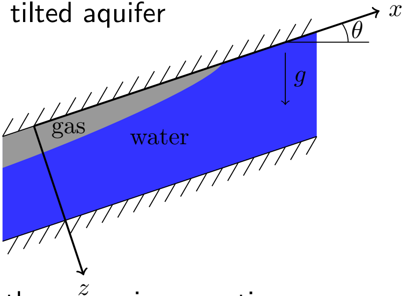

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- **Q21**: Derive the reduced model and fractional flow formulation

$$\frac{\partial \phi s}{\partial t} + \frac{\partial}{\partial x}(f(s)u + g(s)(-H\cos\theta s_x + \sin\theta)) = 0$$

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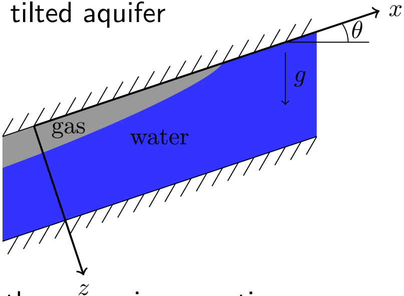

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- **Q22**: Determine the traveling waves.

Project work

- This is an exercise not an evaluation

Project work

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- You do not have to answer to all the questions

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- You can work on questions that are not in the text.

Project work

- This is an exercise not an evaluation
- You do not have to answer to all the questions
- You can work on questions that are not in the text.
- Questions!