TMA 4195 Mathematical Modeling,
30 November 2006
Solution with additional comments

1 Problem

A cookbook states a cooking time of 3.5 hours at the oven temperature T_{oven} for a 3 kg turkey. For a 10 kg (American) turkey, 7 hours is suggested.

(a) How can we derive the heat equation

$$\rho c \frac{\partial T}{\partial t} = k \nabla^2 T, \tag{1}$$

where the density ρ, the heat capacity c, and the heat transmission k, are constant?

We assume that the turkey's heating follows the equation 1. The heat transfer through the turkey's surface follows Newton's law of heating,

$$-k \nabla T|_{\text{surface}} = \beta \left(T_{\text{oven}} - T_{\text{surface}} \right), \tag{2}$$

where β is a constant.

(b) The cooking time, t_s, has to be dependent on, in addiction to the parameters above, the turkey's diameter D, the temperature difference between the oven and the turkey before (ΔT_0) and after (ΔT_1) the cooking (We assume that all turkeys are geometrically similar!). Use dimension analysis to find an expression for t_s (Hint: combine the parameters in Eqns. 1 and 2 before you use them in the dimension analysis).

(c) After short time, the turkey's surface reach the same temperature of the oven. This implies that equation 2 simplifies to $T_{\text{oven}} = T_{\text{surface}}$, and the parameter with β disappears. Use this information to simplify dimension analysis and show that t_s is proportional to the turkey's weight to the power 2/3. How does this match with what has been indicated in the cookbook?

Solution:

(a) The heat density is given by $\rho c T$, and the heat flux is $\mathbf{j} = -k \nabla T$. Without source terms, we obtain the integral conservation law

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_R \rho c T \mathrm{d}V + \int_{\partial R} \mathbf{j} \cdot \mathbf{n} \mathrm{d}\sigma = 0,$$

This implies - by moving the derivative into the integral, using the divergence theorem, and letting R vary - the differential formulation

$$\rho c \frac{\mathrm{d}T}{\mathrm{d}t} + \nabla \cdot (-k \nabla T) = 0.$$

(b) We follow the hint in the Problem and assume that

$$t_s = t_s \left(D, \kappa, \frac{\beta}{k}, \Delta T_0, \Delta T_1 \right),$$

where the heat diffusion coefficient is $\kappa = k / (\rho c)$.

Digression: One could think that T_{surface} and $\nabla T|_{\text{surface}}$ should also be considered. However, these quantities are dependent on time and dependent on the temperature variation. This variation is dependent on β/k and other parameters.

From the equations, we derive that $[\kappa] = \text{s/m}^2$ and $[\beta/k] = \text{m}^{-1}$. Thus, we are able to state the dimension matrix as

	t_s	D	κ	β/k	ΔT_0	ΔT_1
m	0	1	-2	-1	0	0
s	1	0	1	0	0	0
K	0	0	0	0	1	1

The matrix has rank 3 and, hence we have 3 dimensionless combinations which are easily derived:

$$\pi_1 = \frac{\kappa t_s}{D^2},\ \pi_2 = \frac{D\beta}{k},\ \pi_3 = \frac{\Delta T_0}{\Delta T_1},$$

Thus $\pi_1 = \Phi(\pi_2, \pi_3)$, or

$$t_s = \frac{D^2}{\kappa}\Phi\left(\frac{D\beta}{k}, \frac{\Delta T_0}{\Delta T_1}\right).$$

The alternative expression

$$t_s = \frac{1}{\kappa\beta^2}\Psi\left(\frac{D\beta}{k}, \frac{\Delta T_0}{\Delta T_1}\right)$$

is equally correct.

The most elegant solution (found by one of the students) is however to scale the equations 1 and 2 using D for length and t_s for time. Then π_1 and π_2 drop out immediately.

(c) If we get rid of β, we find

$$t_s = \frac{D^2}{\kappa}\Phi\left(\frac{\Delta T_0}{\Delta T_1}\right).$$

Since the weight can be written as $W = G\rho D^3$, where G is a constant dimensionless "geometry factor",

$$t_s = \frac{(W/G\rho)^{2/3}}{\kappa}\Phi\left(\frac{\Delta T_0}{\Delta T_1}\right) \propto W^{2/3}.$$

With cooking time for the 3 kg turkey is 3.5 hours, the cooking time for a 10 kg turkey should be (assuming that ΔT_0 and ΔT_1 remain the same)

$$t_s(10\text{kg}) = t_s(3\text{kg})\left(\frac{10}{3}\right)^{2/3} = 3.5\left(\frac{10}{3}\right)^{2/3} \text{hour} = 7.8 \text{ hour}.$$

Eight hours for a 10 kg turkey would therefore be a better rule than seven hours.

2 Problem

Let us consider the problem

$$\varepsilon\frac{d^2y}{dt^2} + \frac{dy}{dt} = 2t,\ 0 \le t \le 1,\ 0 < \varepsilon \ll 1,$$
$$y(0) = y(1) = 1.$$

What do we call such problems? Find, to leading order in ε, the outer, inner, and uniform solutions to the problem (Hint: the boundary layer is near $t = 0$).

Solution:

Since this is an equation with a small parameter in front of the highest derivative, it is called a *singular perturbation problem.*

The leading order outer solution, $y_0(t)$, is found by setting $\varepsilon = 0$,

$$\frac{dy_0}{dt} = 2t.$$

The general solution is $A + t^2$, but it is impossible to fulfil both the boundary conditions, since $y(0) = 1$ implies that $y_0(t) = 1 + t^2$, and $y(1) = 1$ implies that $y_0(t) = t^2$.

Given the hint, it is reasonable to try a new time scale, $t = \delta\tau$ around 0. This leads to

$$\frac{\varepsilon}{\delta^2} Y_{\tau\tau} + \frac{1}{\delta} Y_\tau = 2\delta\tau.$$

By choosing $\delta = \varepsilon$, the equation becomes

$$Y_{\tau\tau} + Y_\tau = \varepsilon^2 2\tau,$$

with a general solution to the leading order, $Y_0(\tau) = A + Be^{-\tau}$ (One could think that $\delta = \varepsilon^{1/2}$ was a possibility, but this would give an equation $Y_{0\tau} = 0$ which does not help us). If we fulfil the boundary condition $Y_0(0) = 1$, we obtain $A + B = 1$, or

$$Y_0(\tau) = 1 + A\left(e^{-\tau} - 1\right).$$

The solution would satisfy both boundary conditions if we set $A = 0$, but we then apply Y_0 outside its admissible region (this misuse of singular perturbation is sometimes seen in science) .

The matching condition is, in its simplest form, given by

$$\lim_{t\to 0} y_0(t) = \lim_{\tau\to\infty} Y_0(\tau),$$

and this leads to, by using $y_0(1) = 1$,

$$0 = 1 - A,$$

or $A = 1$. Thus, the uniform solution to the leading order is

$$y_0^{(u)} = y_0(t) + Y_0\left(\frac{t}{\varepsilon}\right) - y_0(0) = t^2 + e^{-t/\varepsilon}.$$

The error around $t = 1$ is totally negligible.

Digression (not part of the exam): The exact solution to the equation is

$$y_{ex}(t) = A + Be^{-\frac{t}{\varepsilon}} + t^2 - 2t\varepsilon,$$

and, thus, with a negligible error $\mathcal{O}\left(e^{-1/\varepsilon}\right)$,

$$y_{ex} = t^2 + e^{-t/\varepsilon} + 2\varepsilon\left(1 - e^{-\frac{t}{\varepsilon}} - t\right).$$

Note that "to the leading order in ε" means the $\mathcal{O}(1)$ term and not $\mathcal{O}(\varepsilon)$.

3 Problem

Let $x^*(t^*)$ be the cod population in an ocean region as a function of time t^*. The region may hold a maximum sustainable amount of fish equal to K, and, as long as fishing is prohibited and $x^* \ll K$, x^* will grow with rate r, $dx^*/dt^* = rx^*$. We assume that the amount of caught fish per time unit is $\alpha x^* B$, where B is the number of participating boats , and α is a constant.

(a) State a model for the amount of fish such that, using a suitable scale, we obtain the form

$$\frac{dx}{dt}(t) = x(t) - x^2(t) - \mu x(t).$$

(b) Discuss the equilibrium points for different values of μ. Sketch the possible trajectories path for the amount of fish.

(c) Find an expression for the number of boats giving an optimal management of the fish resources.

Solution:

(a) The equation suggests we should assume a logistic model in absence of fishing activities. Thus, we find immediately that

$$\frac{dx^*}{dt^*} = rx^*\left(1 - \frac{x^*}{K}\right) - \alpha x^* B$$

We scale the model by setting

$$x^* = xK,$$
$$t^* = \frac{1}{r}t$$

Thus,

$$Kr\frac{dx}{dt} = rKx(1-x) - \alpha KxB,$$

or

$$\dot{x} = x(1-x) - \mu x,$$
$$\mu = \frac{\alpha B}{r}.$$

(b) The equilibrium solutions are

$$x_1 = 0,$$
$$x_2 = 1 - \mu.$$

Using straightforward linear stability and

$$\frac{d(x(1-x) - \mu x)}{dx} = 1 - 2x - \mu,$$

we obtain that x_1 is stable for $\mu > 1$ and unstable for $\mu < 1$, while x_2 is stable for $\mu < 1$ and unstable for $\mu > 1$ (the physical acceptable points are clearly x_1, $x_2 \geq 0$).

If $\mu = 1$, the equation becomes

$$\dot{x} = -x^2,$$

and both equilibrium solutions coincide at $x = 0$. This equation can be solved generally:

$$\frac{1}{x} = t + C,$$

i.e.

$$x = \frac{1}{t + C},$$

and *also* $x = 0$, as obviously is a solution. If we start in $x(0) > 0$ and close to 0, we have $x(t) \to 0$ when $t \to \infty$. If, on the contrary and un-physically, $x(0) = 1/C < 0$, we have $x(t) \to -\infty$ when $t \to -C$. The equilibrium point *is* stable for $\mu = 1$, since we have that $x \geq 0$, and the answer is as follows:

$$x_s = \max\{1 - \mu, 0\}, \ \mu \geq 0$$

is stable, while

$$x_u = 0, \ 0 \leq \mu < 1,$$

is unstable.

(c) A constant outtake per time unit can be expressed as

$$Q = (\alpha B)\, x_2^* = (\alpha B)\, K(1 - \mu) = (\mu r)\, K(1 - \mu) = K r \mu (1 - \mu), \tag{3}$$

and this has a maximum for $\mu = 1/2$, that is $B = r/(2\alpha)$, and $x_2^* = K/2$.

4 Problem

In this problem we study the traffic along a road (a one-way street towards $+\infty$ and *without in- or out-flow for the moment). We further assume that all variables are scaled so that the car density* ρ *is between* 0 *and* 1, *and the car speed* v *is* $1 - \rho$.

(a) Show how one derives an expression for the speed U *of a shock in the car density, and that we in this case obtain* $U = 1 - \rho_1 - \rho_2$, *where* ρ_1 *and* ρ_2 *are the densities on each side of the shock.*

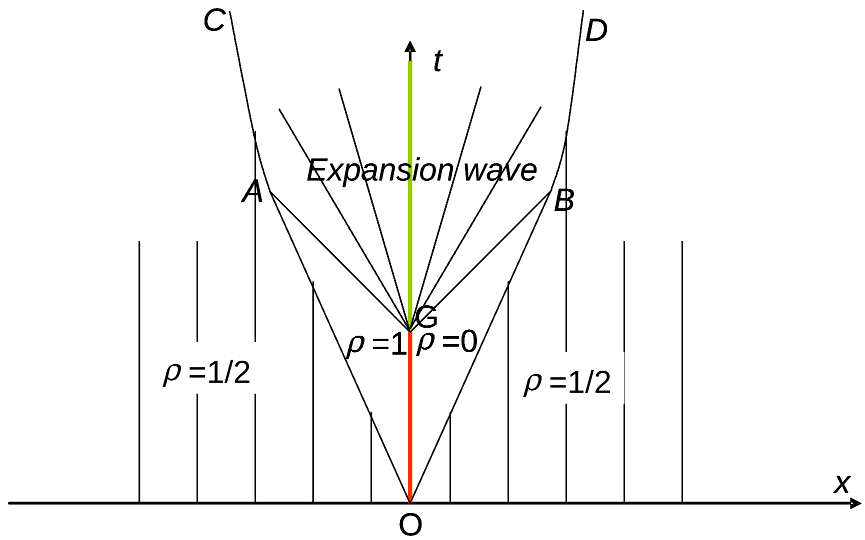
Assume that the car density along the road for $t < 0$ *is equal to* 1/2. *Between* $t = 0$ *and* $t = 1$ *the cars get a red light due to a pedestrian crossing placed in* $x = 0$. *For* $t > 1$, *the light is green.*

(b) Find the solution $\rho(x, t)$ *for* $t \geq 0$. *(Hint: make a sketch of the situation in an* x*-*t*-diagram. Show that the solution for* ρ *has to be determined in 5 different regions, where the values in 4 of them is obvious. In order to find the regions, one has to find their borders).*

A second road (with similar properties as the first one) is now merging from the side with the first one.

(c) Which conservation law has to be fulfilled at the merging point? Assume that the flux on the first road is constant, $j_1 = 1/8$, *and that the density is less than* 1/2. *Describe (without further calculations) the evolution of the car density on the first road when the density* ρ_2 *on the second road increases from* 0 *to* 1. *The cars on the first road are flexible and let entering cars merge whenever it is possible. Consider in particular what happens when the flux on the second road reaches* 1/8.

Solution:



Figur 1: Sketch of the situation around the crossing.

This is the so-called *standard model*: $0 \leq \rho \leq 1,\ v = 1 - \rho,\ j = \rho(1-\rho)$. The conservation law can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b \rho dx + j(b) - j(a) = 0, \tag{4}$$

or in differential form,

$$\rho_t + (\rho(1-\rho))_x = \rho_t + (1-2\rho)\rho_x = 0.$$

(a) We find out the shock speed by considering a shock with speed U in the interval $[a, b]$. The density to the left of the shock is ρ_1, while the one on the right ρ_2. By using the equation 4, we obtain

$$(\rho_2 - \rho_1)U = \rho_2(1-\rho_2) - \rho_1(1-\rho_1),$$

or,

$$U = 1 - \rho_1 - \rho_2.$$

(b) For $\rho = 1/2$ the characteristics are vertical, and we have a situation as sketched in figure 1.

For the shock OA, $\rho_2 = 1$ and $\rho_1 = 1/2$, in other words, the shock speed is $U_{OA} = -1/2$. In an analogous way we find that $U_{OB} = 1/2$, $U_{GB} = 1$ and $U_{GA} = -1$. The point A is at $x = -1$, $t = 2$, and B is at $x = 1$, $t = 2$. Within the rarefaction wave, the solution is given by

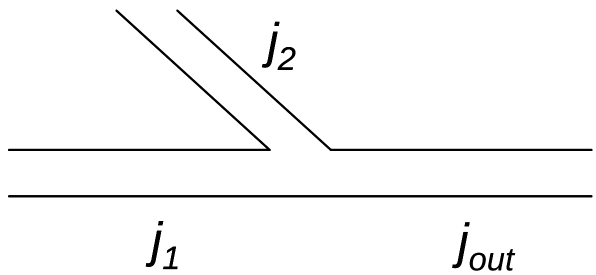
$$x = c(\rho)(t-1) = (1-2\rho)(t-1),$$

or

$$\rho_e(x,t) = \frac{1}{2}\left(1 - \frac{x}{t-1}\right).$$

It remains to find the shocks AC and BD. In a point (x, t) on the shock AC we have that $\rho_1 = 1/2$, while $\rho_2 = \rho_e(x,t) = \frac{1}{2}\left(1 - \frac{x}{t-1}\right)$. Therefore,

$$U(x,t) = 1 - \frac{1}{2} - \left(\frac{1}{2}\left(1 - \frac{x}{t-1}\right)\right) = \frac{x}{2(t-1)}.$$



Figur 2: The second road merges with the first road.

The equation for the shock becomes

$$\dot{x} = \frac{x}{2\,(t-1)},$$
$$x\,(2) = -1.$$

The equation is separable and the solution is $x_{AC}\,(t) = -\sqrt{t-1},\ t \geq 2$. In a equivalent way we find $x_{BD} = \sqrt{t-1}$. Thus, the solution is completely determined.

(c) The situation is sketched in figure 2. Since the crossing is not a parking lot, we must have

$$j_1 + j_2 = j_{\text{out}}.$$

We know that $j_1 = 1/8$, but the equation

$$j_1 = \frac{1}{8} = \rho\,(1-\rho)$$

has to possible solutions,

$$\rho_{1,2} = \frac{1}{2} \pm \sqrt{\frac{1}{8}}.$$

Here, since we also know that $\rho_1 < 1/2$, the density is equal to $\frac{1}{2} - \sqrt{\frac{1}{8}}$.

As long as $j_2 < 1/8$ the flux $j_{\text{out}} < 1/4$, the maximum value.

When j_2 passes 1/8, cars will pile up on the second road in front of the crossing. Since the outflow from the second way can at most be 1/8, the cars' density just before the crossroad becomes

$$\rho_{2f} = \frac{1}{2} + \sqrt{\frac{1}{8}}.$$

In the back of the line we have a shock moving backward with speed

$$U = 1 - \rho_2 - \rho_{2f}.$$

up to $\rho_2 = \rho_{2f}$. When ρ_2 increases further from ρ_{2f}, the flux of the cars on the second road will be so small (<1/8) that all may enter without problems!