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Eksamen i TMA4195 Mathematical Modeling

English
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9:00 – 13:00

Aids (code C): Approved calculator
Rottman: *Matematisk formelsamling*

Results: January 11, 2013

Problem 1 A thin film of water flows down a wall. Its length is much bigger than its thickness d. We want to find a relation for the size of the friction force per area (tangential shear stress) τ by dimensional analysis. We assume that τ depends on d, the kinematic viscosity μ, the speed of the water film along the wall u, and the average size of the unevenness of the wall r. Note that $[\tau]$ = force/area, $[\mu] = \frac{\mathrm{kg}}{\mathrm{m\,s}}$, and $[r]$ = length.

Show that d, u, μ can be used as core variables and find a maximal set of independent dimensionless combinations.

Use the Buckinghams Pi-theorem to show that

$$\tau = \frac{\mu u}{d} C\Big(\frac{d}{r}\Big)$$

where $C(\frac{d}{r})$ is dimensionless.

Problem 2 In a one dimensional chemical reactor of length $L > 0$, the concentration $c^*(x^*, t^*)$ of a chemical substance will satisfy the following reaction-diffusion-convection equation

$$c^*_{t^*} = Dc^*_{x^*x^*} + Uc^*_{x^*} - rc^* \qquad \text{for} \qquad t^* > 0, \quad x^* \in (0, L)$$

where $D, U, r > 0$. It is known that $0 \le c^*(x^*, t^*) \le \max_{x\in\mathbb{R}} c^*(x, 0) =: M > 0$ for $t^* > 0$ and $x^* \in \mathbb{R}$. We consider two different cases:

$$(i)\;\; D \gg UL + rL^2 \qquad \text{and} \qquad (ii)\;\; L^2 r \gg D + UL.$$

Find the natural scaling and scale the equation using this scaling in the two cases.

Problem 3 Two scaled population dynamic models are given by the following two systems of equations:

$$(i)\;\begin{cases} \dfrac{dn_1}{dt} = n_1(1 - n_1 - an_2) \\ \dfrac{dn_2}{dt} = cn_2(1 - n_1) \end{cases} \qquad \text{and} \qquad (ii)\;\begin{cases} \dfrac{dn_1}{dt} = n_1(1 - n_1 + an_2) \\ \dfrac{dn_2}{dt} = cn_2(1 - n_2 + bn_1), \end{cases}$$

where $a, b, c > 0$.

What could these systems model? Hint: Note the signs of the terms in model (ii).

Find the equilibrium points of system (i). What can you say about their stabilties by using linearization?

Problem 4 Consider the following scaled initial value problem

$$\begin{aligned} \dot{x} &= -x + (x+1)(y-1), \quad t > 0, \qquad && x(0) = 1, \\ \epsilon\dot{y} &= x - (x+1)y, \quad t > 0, && y(0) = 0, \end{aligned}$$

where $0 < \epsilon \ll 1$.

Find a uniform/global approximate solution of the problem by perturbation methods.

Hint: The model has two different time scales.

Problem 5 In the standard scaled fluid dynamics model of car traffic along a one-way road, the car density $\rho(x, t)$ will satisfy the equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} j(\rho) = 0,$$

where $j(\rho) = \rho(1 - \rho)$ is the car flux. On the road at $x = 0$ there is a car counter. At $t = 0$ the measured scaled car flux at $x = 0$ increases fast from $\frac{7}{64}$ to $\frac{3}{16}$. We simplify and assume that

$$\begin{aligned} \rho(x, 0) &= \frac{1}{8} \qquad \text{for all} \quad x > 0, \\ j\big(\rho(0, t)\big) &= \frac{3}{16} \qquad \text{for all} \quad t > 0. \end{aligned}$$

Find the characteristics and sketch them in the tx-plane. Justify your answer.

Problem 6 A thin film of water with constant density ρ flows down a wall at a speed u. Its length is much bigger than its thickness d. We introduce a coordinate system in space such that z is the vertical coordinate increasing upwards and the wall corresponds the square $[0,1] \times [0,1]$ in the yz-plane.

We assume that d and u only depend on z and the time t. For $\bar{z} \in (0,1)$, $\bar{t} > 0$, and $\Delta z > 0$ such that $\bar{z} + \Delta z \in (0,1)$, we define the region $R_{\bar{t},\bar{z},\Delta z} \subset \mathbb{R}^3$ as

$$R := R_{\bar{t},\bar{z},\Delta z} = \Big\{(x,y,z) : 0 \le y \le 1,\ 0 \le x \le d(z,\bar{t}),\ \bar{z} \le z \le \bar{z} + \Delta z\Big\}. \tag{1}$$

a) Write down the equation for the conservation of mass in the domain R in integral form.

Show that if d and u are smooth, they will satisfy the following conservation law in differential form,

$$\frac{\partial}{\partial t} d - \frac{\partial}{\partial z}\big(ud\big) = 0 \qquad \text{for all} \qquad t > 0 \quad \text{and} \quad z \in (0,1).$$

Hint: The velocity field is $\vec{v} = -u\vec{e}_z$ where $\vec{e}_z = (0,0,1)$.

The only forces acting on the film of water is gravity and the friction against the wall. That is, the volum and surface denisities of the body and surface forces are respectively

$$\vec{f}_B = -\rho g \vec{e}_z \qquad \text{and} \qquad \vec{f}_S = \begin{cases} C\dfrac{\mu u}{d}\vec{e}_z & \text{if } x = 0 \text{ and } y, z \in (0,1), \\ 0 & \text{otherwise,} \end{cases}$$

where $C > 0$ is a dimensionless constant. These forces are assumed to be at equilibrium,

$$\int_{R_{\bar{t},\bar{z},\Delta z}} \vec{f}_B \, dV = -\int_{\partial R_{\bar{t},\bar{z},\Delta z}} \vec{f}_S \, d\sigma$$

for all $\bar{t} > 0$ and all $\bar{z}$ and $\Delta z > 0$ such that $\bar{z}, \bar{z} + \Delta z \in (0,1)$.

b) Show that if u and d are continuous and $d > 0$, then there is a constant $K > 0$ such that

$$u = K d^2 \qquad \text{for all} \qquad t > 0 \quad \text{and} \quad z \in (0,1).$$

Show that if there are two points $z_1 < z_2$ such that $0 < d(z_1, 0) < d(z_2, 0)$, then the solution d will develop a shock, and the shock will move downwards.

Hint: You do not need to compute the whole solution.