NTNU

Department of Mathematical Sciences

Examination paper for **TMA4195 Mathematical Modeling**

Academic contact during examination: Espen R. Jakobsen

Phone: 91 61 87 27

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Problem 1 The wave speed of water waves can depend the wave length L, the gravitational acceleration g, the density of water ρ, and the surface tension σ. What is the most general form such a (physical) relation may take?

Hint: The surface tension for water is $\sigma = 0{,}07\,\frac{\mathrm{kg}}{\mathrm{s}^2}$.

Problem 2 State the (unscaled) logistic equation and explain all variables and coefficients.

Give the equations for one model involving two interacting populations. Explain what it models.

Problem 3 In a scaled model for a non-linear spring–mass–resistance system, the position of the mass satisfies the following initial-value problem:

$$\ddot{x} + \varepsilon(1 + x^2 + \dot{x}^2)\dot{x} + x = 0, \ t > 0, \qquad x(0) = 1, \qquad \dot{x}(0) = 0,$$

where $0 < \varepsilon \ll 1$, $\dot{x} = \frac{dx}{dt}$ and $\ddot{x} = \frac{d^2x}{dt^2}$.

a) Use perturbation analysis to find an approximate solution whose error is $O(\varepsilon^2)$.

Hint: The method of undetermined coefficients with $At\cos t + Bt\sin t$ may be useful.

In another problem the scaled initial-value problem takes the form:

$$\varepsilon\ddot{x} + (1 + x^2 + \dot{x}^2)\dot{x} + x = 0, \ t > 0, \qquad x(0) = 1, \qquad \dot{x}(0) = 0.$$

b) This problem is equivalent to

$$\begin{cases} \dot{x} = y, & x(0) = 1 \\ \varepsilon\dot{y} = -(1 + x^2 + y^2)y - x, & y(0) = 0. \end{cases} \tag{1}$$

What type of perturbation does this lead to?

Find the inner and outer equations for this system.

Note: You do not need to find the solutions.

Problem 4 We consider car traffic along a one-way one-lane infinite road whithout entrances and exits. Based on measurements of real traffic the engineers propose the following improved model for the scaled flux of cars

$$j(\rho) = \rho - \rho^{\frac{3}{2}}, \tag{2}$$

where ρ is the scaled density of cars, $\rho(x,t) \in [0,1]$, x is the scaled position along the road, and t is scaled time.

a) What is the scaled car velocity in this case?

State without proof the (scaled) conservation law for ρ in differential form.

Show that the solution $\rho(x,t)$ will develope a shock if there are $a < b$ such that

$$\rho(a,0) < \rho(b,0).$$

Note: You do not need to compute the solution.

b) At $x = 0$ there is a traffic light and we consider the case when the light is red for $t < 0$ and green for $t > 0$. To simplify we assume that

$$\rho(x,0) = \begin{cases} 1, & x < 0, \\ 0, & x > 0. \end{cases}$$

Find the car density $\rho(x,t)$ for all $t > 0$ and $x \in \mathbb{R}$.

Problem 5 Consider a long thin tube reactor aligned with the x^*-axis. In the reactor, a chemical C reacts with other chemicals in such a way that the effective production density q^* of C is given by

$$q^* = -rc^*(c^* - a)(c^* - b),$$

where $r > 0$, $0 < a < b$, and c^* is the consentration of C. We also assume that the chemical diffuses and that the diffusion coefficient $D > 0$ is constant.

a) State Fick's law for diffusion, and write down the conservation law for the chemical C in the interval (control volume) $I = [x_1, x_2]$.

Show how you can derive the conservation law in differential form:

$$\frac{\partial c^*}{\partial t^*} = D\frac{\partial^2 c^*}{\partial x^{*2}} - rc^*(c^* - a)(c^* - b), \quad t^* > 0,\ x^* \in \mathbb{R}. \tag{3}$$

b) Show that there is a scaling such that equation (3) in part a) takes the form

$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial x^2} - c(c-k)(c-1), \quad t > 0,\ x \in \mathbb{R}, \tag{4}$$

for some $k \in (0,1)$.

c) Show that the linearization about $c = 0$ of equation (4) is

$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial x^2} - kc, \quad t > 0,\ x \in \mathbb{R}.$$

Solve this equation for any bounded initial data

$$c(x,0) = c_0(x).$$

Hint: The fundamental solution of the heat equation is $c_F(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$.

d) Find all (constant) equilibrium points of equation (4). Determine whether they are stable or not according to linear stability analysis.

Hint: Consider only (small) bounded perturbations. You may use that the fundamental solution $c_F \geq 0$ and $\int_{-\infty}^{\infty} c_F(x,t)dx = 1$.