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Exam in TMA4195 Mathematical Modeling 21.12.2013 Solutions

Problem 1 The dimensional matrix A is

	L	g	ρ	σ	V
kg	0	0	1	1	0
m	1	1	-3	0	1
s	0	-2	0	-2	-1

Since the first 3 collumns are linearly independent, we see that the rank of A is 3 and that we may take L, g, ρ as core variables (they are dimensionally independent). Hence by the first part of Buckingham's Pi Theorem, there are exactly $5 - 3 = 2$ dimensionally independent combinations. The standard choice is now the following:

$$\pi_1 = \frac{\sigma}{L^\bullet g^\bullet \rho^\bullet} \qquad \text{and} \qquad \pi_2 = \frac{V}{L^\bullet g^\bullet \rho^\bullet}.$$

It is then easy to see that

$$\pi_1 = \frac{\sigma}{\rho g L^2} \qquad \text{and} \qquad \pi_2 = \frac{V}{\sqrt{gL}}.$$

The second part of Buckingham's Pi Theorem states that any physical relation

$$\Phi(L, g, \rho, \sigma, V) = 0$$

is equivalent to a relation between the associated dimensionless combinations

$$\Psi(\pi_1, \pi_2) = 0.$$

Hence the most general form a physical relation between L, g, ρ, σ, V may take is

$$\Psi\left(\frac{V}{\sqrt{gL}}, \frac{\rho g L^2}{\sigma}\right) = 0$$

for an arbitrary function Ψ, or if you write V as a function of the other variables,

$$V = \sqrt{gL}\tilde{\Psi}(\frac{\rho g L^2}{\sigma})$$

for an arbitrary function $\tilde{\Psi}$.

Note: Other (equivalent!) relations can be given using other choices of dimensionless combinations.

Problem 2 The unscaled logistic equation is given by

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN(1 - \frac{N}{K}),$$

where N is the population size, r is the growth rate, and K is the carrying capacity.

Possible two-species models are e.g. the predator-prey model (Lotka–Volterra),

$$\begin{aligned}\frac{\mathrm{d}N_1}{\mathrm{d}t} &= r_1 N_1(1 - \alpha N_2),\\ \frac{\mathrm{d}N_2}{\mathrm{d}t} &= r_2 N_2(-1 + \beta N_1),\end{aligned}$$

where N_1 is the prey population and N_2 the predator population (Lotka–Volterra), and the competition model (Gause)

$$\begin{aligned}\frac{\mathrm{d}N_1}{\mathrm{d}t} &= r_1 N_1(1 - \alpha N_2),\\ \frac{\mathrm{d}N_2}{\mathrm{d}t} &= r_2 N_2(1 - \beta N_1),\end{aligned}$$

where N_1 and N_2 are the competing populations (Gause). There are many other possibilities.

Problem 3

a) The perturbation assumption is that

$$x = x_0 + \epsilon x_1 + O(\epsilon^2). \tag{1}$$

From the initial value problem we then get

$$\ddot{x}_0 + \epsilon\ddot{x}_1 + O(\epsilon^2) + \epsilon(1 + x_0^2 + \dot{x}_0^2 + O(\epsilon^2))(\dot{x}_0 + O(\epsilon)) + x_0 + \epsilon x_1 + O(\epsilon^2) = 0, \quad t > 0,$$

and

$$x_0(0) + \epsilon x_1(0) + \ldots = 1 \qquad \text{and} \qquad \dot{x}_0(0) + \epsilon \dot{x}_1(0) + \ldots = 0.$$

Since this should hold for all ϵ, it follows that

$$\begin{aligned} O(1): &\quad \ddot{x}_0 + x_0 = 0, && x_0(0) = 1,\ \dot{x}_0(0) = 0, \\ O(\epsilon): &\quad \ddot{x}_1 + (1 + x_0^2 + \dot{x}_0^2)\dot{x}_0 + x_1 = 0, && x_1(0) = 0,\ \dot{x}_1(0) = 0. \end{aligned}$$

We solve for x_0 and obtain $x_0(t) = \cos t$. It follows that $(1 + x_0^2 + \dot{x}_0^2)\dot{x}_0 = -2\sin t$ and hence the problem for x_1 is the reduced to

$$\ddot{x}_1 + x_1 = 2\sin t, \qquad x_1(0) = 0,\ \dot{x}_1(0) = 0.$$

This is a linear inhomogeneous equation whose general solution can be written as

$$x_1 = x_{hom} + x_{part},$$

where x_{hom} is the general solution of the homogeneous equation $\ddot{x}_1 + x_1 = 0$ and x_{part} is any particular solution of $\ddot{x}_1 + x_1 = -2\sin t$. As for x_0, we find that

$$x_{hom} = C_1 \cos t + C_2 \sin t.$$

To find x_{part}, we use the method of undetermined coefficients and the hint:

$$x_{part} = At\cos t + Bt\sin t.$$

A small computation shows that

$$\ddot{x}_{part} + x_{part} = 2(-A\sin t + B\cos t) = 2\sin t$$

if and only if $A = -1$ and $B = 0$. Taking into account the intitial conditions $x_1(0) = 0 = \dot{x}_1(0)$, we find that $C_1 = 0$ and $C_2 = 1$ respectively, and hence that

$$x_1 = \sin t - t\cos t.$$

The $O(\epsilon^2)$ accurate approximate solution is then

$$x_\epsilon(t) = x_0(t) + \epsilon x_1(t) = \cos t + \epsilon(\sin t - t\cos t).$$

b) First note that the boundary (initial) layer is close to $t = 0$.

The outer equations: Find an approximate solution (x_0, y_0) by setting $\epsilon = 0$ in the original system,

$$\begin{aligned} \dot{x}_0 &= y_0, \\ 0 &= -(1 + x_0^2 + y_0^2)y_0 - x_0. \end{aligned}$$

To find the inner equation, we must rescale and find the boundary (intial) layer thickness which is the other consistent time scale in the problem: $(x, y, t) = (X, Y, \delta\tau)$ and

$$\frac{1}{\delta}\dot{X} = Y,$$
$$\frac{\epsilon}{\delta}\dot{Y} = -(1 + X^2 + Y^2)Y - X.$$

Under the scaling assumption (scaled terms are $O(1)$), balancing in the first equation gives the outer scale $\delta = 1$ while balancing in the second equation gives the inne scale (boundary layer thickness) $\delta = \epsilon$. Let $\delta = \epsilon$, and find approximate solutions (X_0, Y_0) by multiplying the first equation by ϵ and setting $\epsilon = 0$:

$$\dot{X}_0 = 0,$$
$$\dot{Y}_0 = -(1 + X_0^2 + Y_0^2)Y_0 - X_0,$$

(Note that the initial conditions belong to the boundary layer and inner equation, while a maching condition is needed for the outer problem).

Problem 4

a) Here the flux function $j(\rho) = \rho(1 - \sqrt{\rho})$, and since $j(\rho) = v(\rho)\rho$, it follows that $v(\rho) = 1 - \sqrt{\rho}$. The scaled conservation law in differential form is

$$\rho_t + (j(\rho))_x = 0. \tag{2}$$

We look at the characteristics emerging from the points a and b. The characteristic equations ($z(t) = \rho(x(t), t)$) are

$$\dot{x} = c(z) = j'(z), \qquad x(0) = x_0$$
$$\dot{z} = 0, \qquad z(0) = \rho(x(0), 0) = \rho_0(x_0),$$

with x-solution

$$x(t) = x(t; x_0) = x_0 + c(\rho_0(x_0))t.$$

Since $c(\rho) = j'(\rho) = 1 - \frac{3}{2}\rho^{\frac{1}{2}}$ and $\rho(a, 0) < \rho(b, 0)$, it follows that $c(\rho_0(a)) > c(\rho_0(b))$. Hence the characterisitcs to the left ($x(t; a)$) will overtake and collide with the characteristic to the right ($x(t; b)$) in finite time and the solution will develope a shock.

b) The initial value is

$$\rho(x, 0) = \begin{cases} 1 = \rho^-, x < 0, \\ 0 = \rho^+, x > 0, \end{cases}$$

and the method of characteristics gives the speed $c = -\frac{1}{2}$ for characteristics starting at $x_0 < 0$, and $c = 1$ for characteristics starting at $x_0 > 0$. This means that the method of characteristics can not tell us what happens in the sector $-\frac{1}{2}t < x < t$, and we have a rarefaction wave. To find the form of the rarefaction wave solution, we assume that $\rho(x,t) = \phi(\frac{x}{t})$ and insert it into the conservation law. Then

$$-\frac{x}{t^2}\phi' + \frac{1}{t}\phi' j'(\phi) = 0.$$

Divide by ϕ' and use that $j'(\phi) = 1 - \frac{3}{2}\phi^{\frac{1}{2}}$ to get that $\rho(x,t) = \frac{4}{9}(1 - \frac{x}{t})^2$.
The total solution is then

$$\rho(x,t) = \begin{cases} 1, & x < -\frac{1}{2}t, \\ \frac{4}{9}(1-\frac{x}{t})^2, & -\frac{1}{2}t < x < t, \\ 0, & t < x, \end{cases}$$

where the solution outside the sector was found by the method of characteristics.

Problem 5

a) Fick's law: The diffusive flux is proportional to the gradient of the consentration,

$$j_d^* = -D\frac{\partial c^*}{\partial x^*},$$

where D is the diffusion coefficient. The conservation in the interbal I can be stated as

$$\text{Change in no of molcules in } I \text{ per time} = \text{Flux in} - \text{Flux out} + \text{Production},$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1}^{x_2} c^*(x,t^*)\,\mathrm{d}x = j^*(x_1,t^*) - j^*(x_2,t^*) + \int_{x_1}^{x_2} q^*(x,t^*)\,\mathrm{d}x^*$$

If c^* is smooth, then $\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1}^{x_2} c^*(x,t^*)\,\mathrm{d}x = \int_{x_1}^{x_2} \frac{\partial}{\partial t^*}c^*(x,t^*)\,\mathrm{d}x$. Let $x_1 = x_0$ be any point in $\mathbb{R}$ and $x_2 = x_0 + \Delta x$ for $\Delta x > 0$ small, then

$$\frac{1}{\Delta x}\int_{x_0}^{x_0+\Delta x} \frac{\partial}{\partial t^*}c^*(x,t^*)\,\mathrm{d}x = -\frac{j^*(x_0+\Delta x,t^*) - j^*(x_0,t^*)}{\Delta x} + \frac{1}{\Delta x}\int_{x_0}^{x_0+\Delta x} q^*(x,t^*)\,\mathrm{d}x^*.$$

Since Δx is small, $\frac{\partial}{\partial t^*}c^*(x,t^*) \approx \frac{\partial}{\partial t^*}c^*(x_0,t^*)$, $c^*(x,t^*) \approx c^*(x_0,t^*)$, and $q(x,t^*) \approx q(x_0,t^*)$ for $x \in (x, x\Delta x)$. Hence, sending $\Delta x \to 0$, we find that

$$\frac{\partial c^*}{\partial t^*} = -\frac{\partial j^*}{\partial x^*} + q^*$$

at (x_0,t^*). Since this point is arbitrary, the equation holds in $\mathbb{R} \times (0,\infty)$. Inserting for j^* and q^* then gives equation (3) in the problem set.

b) We introduce the scales $c^* = Cc, t^* = Tt, x^* = Xx$. Then

$$\frac{C}{T}c_t = \frac{DC}{X^2}c_{xx} - rC^3c(c - \frac{a}{C})(c - \frac{b}{C}).$$

If we take $C = b$ and divide by $\frac{C}{T}$ we get

$$c_t = \frac{TD}{X^2}c_{xx} - Trb^2c(c - \frac{a}{b})(c - 1).$$

Equation (4) in the problem set now follows from first taking T such that $Trb^2 = 1$ and then X such that $\frac{TD}{X^2} = 1$. Note that then $k = \frac{a}{b} < 1$.

c) We linearize the equation around $c = 0$. It is more transparent to write the equation as

$$c_t = c_{xx} + q(c).$$

The only term we need to linearize is q, since the other terms are already linear:

$$q(c) \approx q(0) + q'(0)c = 0 - kc,$$

and hence the linearized equation is

$$c_t = c_{xx} - kc.$$

We can solve the linearized equation in many ways, e.g. we can transform it to the heat equation using an integrating factor. Let $v = e^{kt}c$ and note that

$$\frac{\partial}{\partial t}v = e^{kt}(c_t + kc) = e^{kt}c_{xx} = v_{xx}.$$

Hence v can be given in terms of the fundamental solution c_F, $v = c_F(\cdot, t) * u_0$, and

$$c(x,t) = e^{-kt}v = e^{-kt}\int_{-\infty}^{+\infty} u_0(y)\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}\,\mathrm{d}y.$$

d) The equilibrium points of (4) are its constant solutions, and if $c = c_e$ is a constant solution of (4), then $(c_e)_t = (c_e)_{xx} = 0$ and

$$q(c_e) = -c_e(c_e - k)(c_e - 1) = 0.$$

The solutions/equilibrium points are $c_e = 0$, $c_e = k$, and $c_e = 1$.

To study the stability of the equilibrium points c_e, we check whether solutions of the equation linearized arond c_e that start near c_e remain near for all times. To do that, let

$$c(x,t) = c_e + \tilde{c}(x,t)$$

and note that if $\tilde{c}$ is not so big, then

$$\tilde{c}_t = \tilde{c}_{xx} + q(c_e + \tilde{c}) \approx \tilde{c}_{xx} + q(c_e) + q'(c_e)\tilde{c}.$$

Note that $q(c_e) = 0$ and let $\hat{c}$ be the solution of the linearized equation

$$c_t = c_{xx} + q'(c_e)c. \tag{5}$$

This linearized equation only has the equilibrium point $\hat{c} = 0$ (since $q' \neq 0$). By definition we say that c_e is a stable (unstable) equilibrium point of the original non-linear equation according to linear stability analysis if $\hat{c} = 0$ is a stable (unstable) equilibrium point of the linearized equation (5).

We solve equation (5) and $c(x, 0) = c_0(x)$ as in part c), this time with using the integrating factor $e^{-q'(c_e)t}$:

$$\hat{c}(x, t) = e^{q'(c_e)t} \int_{\mathbb{R}} c_0(x - y)c_F(y, t)dy.$$

Note that if $|c_0 - 0| = |c_0| < \delta$, then

$$|\hat{c}(x, t) - 0| = |\hat{c}(x, t)| \leq e^{q'(c_e)t} \int_{\mathbb{R}} |c_0(x - y)c_F(y, t)|dy < \delta e^{q'(c_e)t},$$

since by the hint, $\int_{\mathbb{R}} |c_F| = \int_{\mathbb{R}} c_F = 1$. Hence it follows that $\hat{c} = 0$ is a stable equilibrium point if $q'(c_e) \leq 0$ since then small perturbations remain small for all times. On the other hand, if $q'(c_e) > 0$, then $\hat{c} = 0$ is not stable any more since we can find small perturbations that blows up in time. Take e.g. $c_0 = \delta$ and check that

$$\hat{c}(x, t) = \delta e^{-q'(c_e)t} \to \infty \qquad \text{as} \qquad t \to \infty.$$

We compute q' and find that $q'(0) = -k < 0$, $q'(1) = -(1-k) < 0$, and $q'(k) = k(1-k) > 0$ since $0 < k < 1$. From the discussion above we can then conclude according to linear stability analysis that $c_e = 0$ and $c_e = 1$ are stable while $c_e = k$ is unstable.