Norwegian University of
Science and Technology

Department of Mathematical Sciences

Examination paper for **TMA4195 Mathematical Modeling**

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Problem 1 Use the Buckingham's Pi theorem to show that the period of a pendulum must be independent of its mass.

Problem 2 The equation

$$m\frac{d^2y^*}{dt^{*2}} + b\frac{dy^*}{dt^*} + ky^* = 0$$

models a mechanical system consisting of a spring with damping, where $m, b, k > 0$. Explain which possible scales we have for t^* in this model. Provide a suitable scaling of the equation for the case when the two first terms dominate.

Problem 3 In the scaled fluid dynamics model of car traffic along a one-way one-lane road the density of cars ρ satisfies the equation

$$\rho_t + (1 - 2\rho)\rho_x = 0$$

in any x-interval where no cars can enter or leave the road. Define the car flux and speed for this model. At $x = 0$ there is a traffic light, sketch the characteristics and the solution $\rho(x, t)$ for $t > 0$ in the following situation.

The initial condition is

$$\rho(x, 0) = \begin{cases} 1 & x < 0, \\ 0 & x \geq 0. \end{cases}$$

Which kind of car traffic situation could be modelled with this initial density?

Problem 4 This exercise is about the kinetic theory of flood waves in rivers. The starting point is the shallow water equations

$$\begin{aligned} h_t + (vh)_x &= 0, & (1) \\ (vh)_t + (v^2h + \frac{1}{2}gh^2)_x &= gh\sin\alpha - C_f v^2. & (2) \end{aligned}$$

We model the flow of water in a river flowing downhill with a inclined bottom forming an angle α with respect to a reference frame.

Here $h(x, t)$ is the water depth, $v(x, t)$ is the velocity component in the x direction, g is gravity, α is the angle defining the slope of the bottom (water flowing downhill), and C_f is a friction coefficient modeling the roughness of the bottom. The acting forces are hydrostatic pressure, gravity and friction forces.

a) Assume that the gravity force is about of the same size of the force due to friction and they dominate, i.e. the left hand side of the momentum equation is nearly zero. Derive a single conservation law for h.

b) Show that if for the initial water depth h_0 we have that

$$\left.\frac{\partial h_0(x)}{\partial x}\right|_{x=x_0} < 0$$

at some point x_0, then a shock is formed. Compute the speed of the shock and show that the speed of the shock is at least 50% higher than the speed of water downstream.

Problem 5 In this exercise we will derive the equations of motion of the Kapitza pendulum and analyse their equilibria, see Figure 1. The Kapitza pendulum is a simple pendulum undergoing a vertical oscillation of small amplitude and high frequency at the pivot. A simple pendulum has two equilibrium configurations stable down and unstable up. By applying a vertical oscillation of small amplitude and high frequency at the pivot, the inverted state of the pendulum becomes stable (for appropriately high frequencies and small amplitudes).

Denote by

l	the length of the pendulum,
m	the mass of the pendulum,
g	the acceleration due to gravity.

The coordinates of the bob[1] are $(x(t), y(t))$ and written in terms of the angle $\varphi(t)$ they satisfy

$$\begin{aligned} x(t) &= l\,\sin(\varphi) && (3)\\ y(t) &= l\,\cos(\varphi) + a\cos(\nu t), && (4) \end{aligned}$$

where $a\cos(\nu t)$ is the displacement of the vibrating pivot (amplitude a and frequency ν). A part from the vibration effect, which is already included in the coordinate $y(t)$ in (4), we assume that the only acting force is gravity $F_g = mg\,\mathbf{e}_2$, where $\mathbf{e}_2$ is the vector with components $\mathbf{e}_2 = [0, 1]^T$.

[1]The bob of a pendulum is the weight attached at the tip of the pendulum.

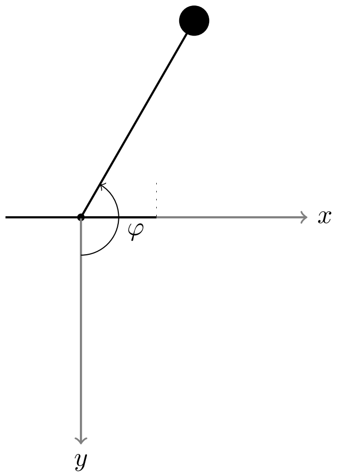

Figure 1: Kapitza pendulum. Pivot in the origin of the coordinate axis. The bob is attached to a rigid rod of length l. A harmonic vertical displacement is applied at the pivot to sabilise the pendulum at the unstable equilibrium at $\varphi = \pi$.

a) Use Newton second law and show that the equation of motion for the Kapitza pendulum is

$$\ddot{\varphi} + \left(\frac{a\nu^2}{l} \cos(\nu t) + \frac{g}{l} \right) \sin(\varphi) = 0. \tag{5}$$

b) (This is the most difficult exercise of the exam).

Laboratory experiments show that the stabilization of the inverted pendulum by applying an oscillatory displacement to the pivot occurs for small enough amplitudes a and big enough frequencies ν, i.e. when $a \to 0$ and $\nu \to \infty$. This is the regime we are interested in.

Assume that the solution φ can be written as a sum of a smooth part θ and a small and highly oscillatory part δ, so that

$$\varphi = \theta + \delta,$$

and assume that the rapid oscillation has the form

$$\delta = \frac{a}{l} \sin(\theta) \cos(\nu t), \tag{6}$$

and when $a \to 0$ then $\delta \to 0$. Derive the equation which θ should satisfy. Write such equation as an expansion in powers of δ, where you include only first order terms in δ.

The period of δ is $\frac{2\pi}{\nu}$, over one such period θ varies very slowly and can be considered constant. Show that using (6) and integrating the equation over the time period $[t, t + \frac{2\pi}{\nu}]$ one obtains the following equation for θ

$$\ddot{\theta} = -\left(\frac{g}{l}\sin\theta + \frac{1}{2}\frac{a^2\nu^2}{l^2}\sin\theta\cos\theta\right). \tag{7}$$

See the appendix for an explanation of why the solution of (7) is a good approximation of $\varphi - \delta$.

c) We now note that equation (7) can be written as

$$\ddot{\theta} = -\frac{\partial U(\theta)}{\partial \theta},$$

for some appropriate potential energy function $U(\theta)$. Write the equation as a system for the variables θ and $v = \dot{\theta}$. Find the total energy $E(\theta) = K(\dot{\theta}) + U(\theta)$ which is conserved along solutions of this system.

Show that $\theta = \pi$, $\dot{\theta} = 0$ is a stable equilibrium. (You might use the Liapunov theorem for proving stability of the equilibrium, see appendix.)

Averaging

Given a differential equation depending on a small parameter ε

$$\dot{x} = \varepsilon f(x, t, \varepsilon), \quad x(0) = x_0, \quad x, x_0 \in D \subset \mathbf{R}^n$$

with periodic solution with period T the averaged equation is the equation

$$\dot{z} = \varepsilon \bar{f}(z), \quad z(0) = z_0$$

with

$$\bar{f}(x) := \frac{1}{T} \int_0^T f(x, s, 0)\, ds.$$

It can be shown that for time intervals of size $\frac{1}{\varepsilon}$, $z(t)$ is an approximation of $x(t)$ of order ε. In Figure 2 you can see an illustration of the effect of averaging. Notice that for a second order equation one can apply the same technique by first rewriting it as a first order system.

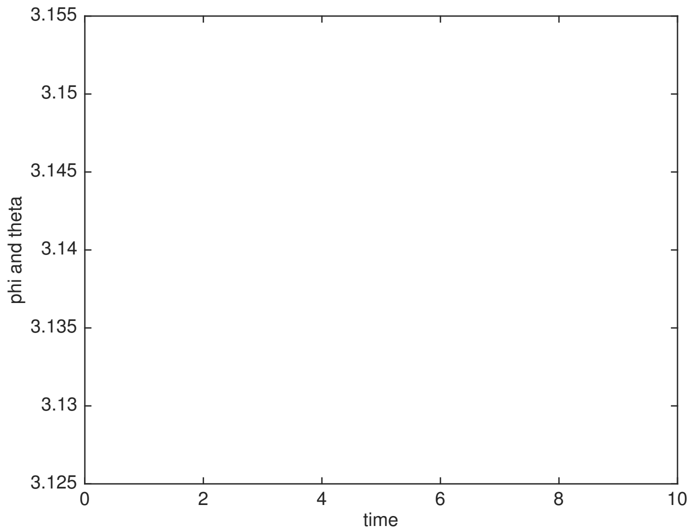

Figure 2: Comparison of the solution of the solution of the Kapitza pendulum (5) dotted line, and of the corresponding averaged equation (7) solid line. In this numerical test $a = 0.1$, $\nu = 50$, $l = 1$ and initial value is $\pi + 0.01$. The Kapitza pendulum oscillates around the equilibrium at π.

Liapunov stability

Theorem 1 *Let y_e be an equilibrium for $\dot{y} = F(y)$. Let $L : \mathcal{O} \to \mathbf{R}$ be differentiable, and let $\mathcal{O} \subset \mathbf{R}^n$ be an open set such that $y_e \in \mathcal{O}$.*

Suppose

(a) $L(y_e) = 0$, $L(y) > 0$ *for* $y \neq y_e$,

(b) $\frac{d}{dt} L \leq 0$ *in* $\mathcal{O} - \{y_e\}$,

then y_e is stable.