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Department of Mathematical Sciences

Examination paper for **TMA4195 Mathematical Modeling**

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Problem 1 Use the Buckingham's Pi theorem to show that the period of a pendulum must be independent of its mass.

Solution We can assume that the physical quantities involved are:

- the velocity v;
- the length of the pendulum ℓ;
- the mass of the bob M;
- acceleration of gravity g.

We have a physical relation

$$T = T(v, \ell, M, g).$$

The physical quantities involved give a dimension matrix

	v	ℓ	M	g	T
kg	0	0	1	0	0
m	1	1	0	1	0
s	-1	0	0	-2	1

We see that the dimension matrix has rank $r = 3$ (because for example the second, third and fifth columns are clearly linearly independent). There are $n = 5$ physical quantities and by the Buckingham's Pi theorem there are $n - r = 2$ independent dimensionless combinations in the equation of this model. We can take these to be

$$\pi_1 = g\frac{T^2}{\ell}, \qquad \pi_2 = v\frac{T}{\ell}.$$

We notice that M cannot be used do make dimensionless combinations because the unit kg does not appear in any of the other involved physical quantities. By the Buckingham's Pi theorem

$$T = T(v, \ell, M, g) \Leftrightarrow \phi(\pi_1, \pi_2) = 0.$$

So for example

$$\pi_1 = \varphi(\pi_2) \Leftrightarrow g\frac{T^2}{\ell} = \varphi\left(v\frac{T}{\ell}\right), \Leftrightarrow T = \sqrt{\frac{\ell}{g}}\,\sqrt{\varphi\left(v\frac{T}{\ell}\right)},$$

and the obtained relation for T is independent on M.

If for simplicity we assume $\sqrt{\varphi\left(v\frac{T}{\ell}\right)} = C$ then

$$T = C\sqrt{\frac{\ell}{g}}$$

and we observe that with $C = 2\pi$, this is the period of a pendulum for small oscillations whose equation is $\ddot{\theta} = -\frac{g}{\ell}\theta$, and solution $\theta(t) = \theta_0\,\cos(\sqrt{\frac{g}{\ell}}t)$.

Problem 2 The equation

$$m\frac{d^2y^*}{dt^{*2}} + b\frac{dy^*}{dt^*} + ky^* = 0$$

models a mechanical system consisting of a spring with damping, where $m, b, k > 0$. Explain which possible scales we have for t^* in this model. Provide a suitable scaling of the equation for the case when the two first terms dominate.

Solution: We scale y^* and t^* by:

$$y^* = Yy, \qquad t^* = Tt.$$

We get the scaled equation

$$\frac{mY}{T^2}\frac{d^2y}{dt^2} + \frac{bY}{T}\frac{dy}{dt} + kYy = 0.$$

The possible scales correspond to the following three situations

i) the first two terms balance and dominate:

$$\frac{mY}{T^2} = \frac{bY}{T} \Rightarrow T = \frac{m}{b};$$

this is valid for $\frac{k}{m}\frac{m^2}{b} = \frac{km}{b} << 1$.

ii) the first and last term balance and they dominate:

$$\frac{mY}{T^2} = kY \Rightarrow T = \sqrt{\frac{m}{k}};$$

this is valid for $\frac{bT}{m} = \frac{b}{m\sqrt{\frac{m}{k}}} = \frac{b}{\sqrt{m}}\frac{1}{k} << 1$.

iii) the last two terms balance and dominate

$$\frac{bY}{T} = kY \Rightarrow T = \frac{b}{k},$$

valid when $m \approx b \approx k \approx 1$.

We get three different scales for T for the three different situations.

If the first and the second term balance and they dominate over the third term then

$$\frac{mY}{T^2} = \frac{bY}{T} \Rightarrow T = \frac{m}{b},$$

and

$$\frac{bY}{T} \geq kY \Rightarrow \frac{b^2}{m}Y \geq kY,$$

with $\frac{km}{b^2}$ small. So if the first two terms dominate, we divide the equations by $\frac{b^2}{m}$ and get

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + \epsilon y = 0,$$

where $\epsilon = \frac{km}{b^2}$.

Problem 3 In the scaled fluid dynamics model of car traffic along a one-way one-lane road the density of cars ρ satisfies the equation

$$\rho_t + (1 - 2\rho)\rho_x = 0$$

in any x-interval where no cars can enter or leave the road. Define the car flux and speed for this model. At $x = 0$ there is a traffic light, sketch the characteristics and the solution $\rho(x, t)$ for $t > 0$ in the following situation.

The initial condition is

$$\rho(x, 0) = \begin{cases} 1 & x < 0, \\ 0 & x \geq 0. \end{cases}$$

Which kind of car traffic situation could be modelled with this initial density? **Solution**: The flux is $j(\rho) = \rho\, v = \rho(1 - \rho)$ and the speed is $v = (1 - \rho)$. The characteristics are the solutions of

$$\begin{aligned} \dot{x} &= 1 - 2z, \quad & x(0) = x_0 \\ \dot{z} &= 0, & z(0) = \rho_0(x_0) \end{aligned}$$

and so

$$z(t) = \rho_0(x_0) = \begin{cases} 1 & x_0 < 0, \\ 0 & x_0 \geq 0, \end{cases}$$

$$x = x_0 + t \begin{cases} -1 & x_0 < 0, \\ 1 & x_0 \geq 0. \end{cases}$$

The characteristics are straight lines parallel to $x = -t$ in the left half plane and parallel to $x = t$ on the right half plane. There is a dead sector (between $x = -t$ and $x = t$) and a rarefaction wave arises. The solution is

$$\rho(x,t) = \begin{cases} 1 & x < -t, \\ \varphi(\frac{x}{t}) & -t < x < t \\ 0 & x \geq t. \end{cases}$$

where $\varphi(\frac{x}{t})$ must satisfy $\rho_t + (j(\rho))_x = 0$ with $j(\rho) = \rho(1-\rho)$. So inserting $\rho = \varphi(\frac{x}{t})$ into this equation and differentiating one obtains

$$j'\left(\varphi\left(\frac{x}{t}\right)\right) = \frac{x}{t}$$

and

$$1 - 2\varphi\left(\frac{x}{t}\right) = \frac{x}{t}$$

leading to

$$\varphi\left(\frac{x}{t}\right) = \frac{1}{2} - \frac{x}{2t}.$$

So

$$\rho(x,t) = \begin{cases} 1 & x < -t, \\ \frac{1}{2} - \frac{x}{2t} & -t < x < t \\ 0 & x \geq t. \end{cases}$$

The car traffic situation described is "red to green light at $x = 0$ and $t = 0$".

Problem 4 This exercise is about the kinetic theory of flood waves in rivers. The starting point is the shallow water equations

$$\begin{aligned} h_t + (vh)_x &= 0, & (1) \\ (vh)_t + (v^2 h + \frac{1}{2} g h^2)_x &= gh \sin\alpha - C_f v^2. & (2) \end{aligned}$$

We model the flow of water in a river flowing downhill with a inclined bottom forming an angle α with respect to a reference frame.

Here $h(x,t)$ is the water depth, $v(x,t)$ is the velocity component in the x direction, g is gravity, α is the angle defining the slope of the bottom (water flowing downhill), and C_f is a friction coefficient modeling the roughness of the bottom. The acting forces are hydrostatic pressure, gravity and friction forces.

a) Assume that the gravity force is about of the same size of the force due to friction and they dominate, i.e. the left hand side of the momentum equation is nearly zero. Derive a single conservation law for h.

Solution: By the assumption that the gravity force is about of the same size of the force due to friction and they are much bigger than the forces due to acceleration and pressure, the left hand side is nearly zero and we get

$$gh\sin(\alpha) = C_f v^2 \Rightarrow v = \sqrt{\frac{g\sin(\alpha)}{C_f}}\, h^{\frac{1}{2}} = Kh^{\frac{1}{2}}, \quad K := \sqrt{\frac{g\sin(\alpha)}{C_f}}.$$

Inserting the expression for v into (1) we get

$$h_t + \frac{\partial}{\partial x} j(h) = 0, \qquad j(h) := K\, h^{\frac{3}{2}}.$$

b) Show that if for the initial water depth h_0 we have that

$$\left.\frac{\partial h_0(x)}{\partial x}\right|_{x=x_0} < 0$$

at some point x_0, then a shock is formed. Compute the speed of the shock and show that the speed of the shock is at least 50% higher than the speed of water downstream.

Solution: We have a shock if

$$\frac{\partial}{\partial x}(c(h_0)) = \frac{\partial}{\partial x}(j'(h_0)) = \frac{\partial}{\partial x}(\frac{3}{2}Kh_0^{\frac{1}{2}}) = \frac{3}{4}Kh_0^{-\frac{1}{2}}\,(h_0)_x < 0.$$

So we have a shock if

$$(h_0)_x < 0$$

or in other words if the initial water depth is decreasing in the x direction. The shock speed is

$$\dot{s} = K\,\frac{h_l^{\frac{3}{2}} - h_r^{\frac{3}{2}}}{h_l - h_r} \geq K \min_{h\in[h_r,h_l]} \frac{\partial}{\partial h} h^{\frac{3}{2}} \geq Kh_r^{\frac{1}{2}} = \frac{3}{2}v_r,$$

where $v_r = Kh_r^{\frac{1}{2}}$ is the velocity downstream.

Problem 5 In this exercise we will derive the equations of motion of the Kapitza pendulum and analyse their equilibria, see Figure 1. The Kapitza pendulum is a simple pendulum undergoing a vertical oscillation of small amplitude and high frequency at the pivot. A simple pendulum has two equilibrium configurations stable down and unstable up. By applying a vertical oscillation of small amplitude and high frequency at the pivot, the inverted state of the pendulum becomes stable (for appropriately high frequencies and small amplitudes).

Denote by

l	the length of the pendulum,
m	the mass of the pendulum,
g	the acceleration due to gravity.

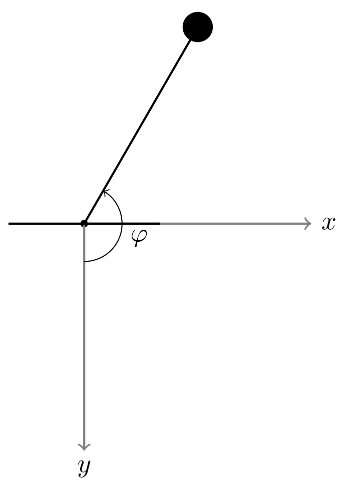

Figure 1: Kapitza pendulum. Pivot in the origin of the coordinate axis. The bob is attached to a rigid rod of length l. A harmonic vertical displacement is applied at the pivot to sabilise the pendulum at the unstable equilibrium at $\varphi = \pi$.

The coordinates of the bob[1] are $(x(t), y(t))$ and written in terms of the angle $\varphi(t)$ they satisfy

$$x(t) = l\,\sin(\varphi) \tag{3}$$

$$y(t) = l\,\cos(\varphi) + a\cos(\nu t), \tag{4}$$

where $a\cos(\nu t)$ is the displacement of the vibrating pivot (amplitude a and frequency ν). A part from the vibration effect, which is already included in the

[1]The bob of a pendulum is the weight attached at the tip of the pendulum.

coordinate $y(t)$ in (4), we assume that the only acting force is gravity $F_g = mg\,\mathbf{e}_2$, where $\mathbf{e}_2$ is the vector with components $\mathbf{e}_2 = [0, 1]^T$.

a) Use Newton second law and show that the equation of motion for the Kapitza pendulum is

$$\ddot{\varphi} + \left(\frac{a\nu^2}{l}\cos(\nu t) + \frac{g}{l}\right)\sin(\varphi) = 0. \tag{5}$$

Solution: Differentiating x and y with respect to t twice we obtain the components of the acceleration

$$\begin{aligned}\ddot{x} &= -l\,\sin(\varphi)\,\dot{\varphi}^2 + l\,\cos(\varphi)\,\ddot{\varphi}\\ \ddot{y} &= -l\,\cos(\varphi)\,\dot{\varphi}^2 - l\,\sin(\varphi)\,\ddot{\varphi} - a\nu^2\cos(\nu t).\end{aligned}$$

By Newton second law and since the only acting force is gravity, we get the equations

$$\begin{bmatrix} m\ddot{x} \\ m\ddot{y} \end{bmatrix} = \begin{bmatrix} 0 \\ m\,g \end{bmatrix},$$

leading to

$$\begin{aligned}\dot{\varphi}^2 &= \frac{\cos(\varphi)}{\sin(\varphi)}\ddot{\varphi},\\ \frac{\cos(\varphi)^2}{\sin(\varphi)}\ddot{\varphi} + \sin(\varphi)\ddot{\varphi} + \frac{a\nu^2}{l}\cos(\nu t) &= -\frac{g}{l},\end{aligned}$$

and to

$$\ddot{\varphi} = -\left(\frac{g}{l} + \frac{a\nu^2}{l}\cos(\nu t)\right)\sin(\varphi).$$

b) (This is the most difficult exercise of the exam).

Laboratory experiments show that the stabilization of the inverted pendulum by applying an oscillatory displacement to the pivot occurs for small enough amplitudes a and big enough frequencies ν, i.e. when $a \to 0$ and $\nu \to \infty$. This is the regime we are interested in.

Assume that the solution φ can be written as a sum of a smooth part θ and a small and highly oscillatory part δ, so that

$$\varphi = \theta + \delta,$$

and assume that the rapid oscillation has the form

$$\delta = \frac{a}{l}\sin(\theta)\cos(\nu t), \tag{6}$$

and when $a \to 0$ then $\delta \to 0$. Derive the equation which θ should satisfy. Write such equation as an expansion in powers of δ, where you include only first order terms in δ.

The period of δ is $\frac{2\pi}{\nu}$, over one such period θ varies very slowly and can be considered constant. Show that using (6) and integrating the equation over the time period $[t, t+\frac{2\pi}{\nu}]$ one obtains the following equation for θ

$$\ddot{\theta} = -\left(\frac{g}{l}\sin\theta + \frac{1}{2}\frac{a^2\nu^2}{l^2}\sin\theta\cos\theta\right). \tag{7}$$

See the appendix for an explanation of why the solution of (7) is a good approximation of $\varphi - \delta$.

Solution: Differentiating twice with respect to t the expression for δ we obtain

$$\ddot{\delta} = -\frac{a}{l}\cos(\nu t)\sin(\theta)\dot{\theta}^2 + \frac{a}{l}\cos(\theta)\cos(\nu t)\ddot{\theta} - 2\frac{a}{l}\nu\cos(\theta)\sin(\nu t)\dot{\theta} - \frac{a}{l}\nu^2\sin(\theta)\cos(\nu t).$$

From this we obtain

$$\begin{aligned}
\ddot{\theta} = \ddot{\varphi} - \ddot{\delta} \;&=\; -\left(\frac{g}{l} + \frac{a\nu^2}{l}\cos(\nu t)\right)\left(\sin(\theta) + \delta\cos(\theta) + \mathcal{O}(\delta^2)\right) \\
&- \;\; \left(\frac{a}{l}\cos(\nu t)\sin(\theta)\dot{\theta}^2 - \frac{a}{l}\cos(\theta)\cos(\nu t)\ddot{\theta} + \right. \\
&\quad \left. -2\frac{a}{l}\nu\cos(\theta)\sin(\nu t)\dot{\theta} - \frac{a}{l}\nu^2\sin(\theta)\cos(\nu t)\right),
\end{aligned}$$

$$\begin{aligned}
\ddot{\theta} \;&=\; -\frac{g}{l}\sin(\theta) - \left(\frac{g}{l} + \frac{a\nu^2}{l}\cos(\nu t)\right)\delta\cos(\theta) + \mathcal{O}(\delta^2) \\
&- \left(\frac{a}{l}\cos(\nu t)\sin(\theta)\dot{\theta}^2 - \frac{a}{l}\cos(\theta)\cos(\nu t)\ddot{\theta} + \right. \\
&\left. -2\frac{a}{l}\nu\cos(\theta)\sin(\nu t)\dot{\theta}\right),
\end{aligned}$$

Substituting for δ from (6) we finally have

$$\begin{aligned}
\ddot{\theta} \;&=\; -\frac{g}{l}\sin(\theta) - \left(\frac{g}{l} + \frac{a\nu^2}{l}\cos(\nu t)\right)\frac{a}{l}\sin(\theta)\cos(\nu t)\cos(\theta) + \mathcal{O}(\delta^2) \\
&- \left(\frac{a}{l}\cos(\nu t)\sin(\theta)\dot{\theta}^2 - \frac{a}{l}\cos(\theta)\cos(\nu t)\ddot{\theta} + \right. \\
&\left. -2\frac{a}{l}\nu\cos(\theta)\sin(\nu t)\dot{\theta}\right),
\end{aligned}$$

Averaging over one period $[t, t+\frac{2\pi}{\nu}]$ the obtained equation (while keeping θ constant, since θ varies slowly compared to δ over such interval of time), and dropping terms of higher order in δ, we see that all terms which are multiplied by one of the integrals

$$\frac{\nu}{2\pi}\int_t^{t+\frac{2\pi}{\nu}} \cos(\nu s)ds = 0, \qquad \frac{\nu}{2\pi}\int_t^{t+\frac{2\pi}{\nu}} \sin(\nu s)ds = 0,$$

become zero, while there is one term where the factor

$$\frac{\nu}{2\pi}\int_t^{t+\frac{2\pi}{\nu}} \cos(\nu s)^2 ds = \frac{1}{2}$$

appears, so all together we are left with

$$\ddot{\theta} = -\left(\frac{g}{l}\sin\theta + \frac{1}{2}\frac{a^2\nu^2}{l^2}\sin\theta\cos\theta\right).$$

c) We now note that equation (7) can be written as

$$\ddot{\theta} = -\frac{\partial U(\theta)}{\partial\theta},$$

for some appropriate potential energy function $U(\theta)$. Write the equation as a system for the variables θ and $v = \dot{\theta}$. Find the total energy $E(\theta) = K(\dot{\theta}) + U(\theta)$ which is conserved along solutions of this system.

Show that $\theta = \pi$, $\dot{\theta} = 0$ is a stable equilibrium. (You might use the Liapunov theorem for proving stability of the equilibrium, see appendix.)

Solution: The system is

$$\begin{aligned}\dot{\theta} &= v\\ \dot{v} &= -\frac{\partial U}{\partial\theta},\end{aligned}$$

where

$$U(\theta) = -\frac{g}{l}\cos(\theta) + \frac{1}{4}\frac{a^2\nu^2}{l^2}\sin(\theta)^2.$$

The total energy $E(\theta, v) = \frac{1}{2}v^2 + U(\theta)$ is preserved along solutions of the system, in fact

$$\frac{dE}{dt} = v\dot{v} - \frac{\partial U}{\partial\theta}v = 0.$$

We see that $(0, \pi)$ is an equilibrium of the system and we use the Liapunov theorem given in the appendix to show stability.

We consider the Liapunov function

$$L(\theta, v) := E(\theta, v) - E(\pi, 0).$$

This function satisfies the hypothesis of the theorem and therefore the equilibrium is stable.

Averaging

Given a differential equation depending on a small parameter ε

$$\dot{x} = \varepsilon f(x, t, \varepsilon), \quad x(0) = x_0, \quad x, x_0 \in D \subset \mathbf{R}^n$$

with periodic solution with period T the averaged equation is the equation

$$\dot{z} = \varepsilon \bar{f}(z), \quad z(0) = z_0$$

with

$$\bar{f}(x) := \frac{1}{T} \int_0^T f(x, s, 0)\, ds.$$

It can be shown that for time intervals of size $\frac{1}{\varepsilon}$, $z(t)$ is an approximation of $x(t)$ of order ε. In Figure 2 you can see an illustration of the effect of averaging. Notice that for a second order equation one can apply the same technique by first rewriting it as a first order system.

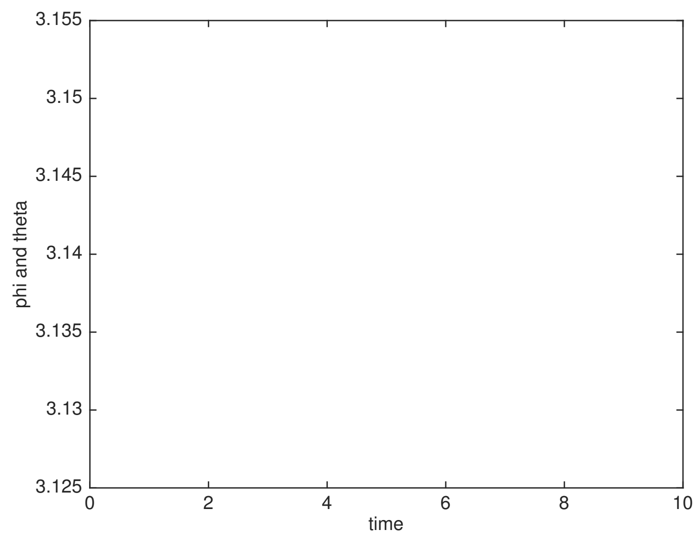

Figure 2: Comparison of the solution of the solution of the Kapitza pendulum (5) dotted line, and of the corresponding averaged equation (7) solid line. In this numerical test $a = 0.1$, $\nu = 50$, $l = 1$ and initial value is $\pi + 0.01$. The Kapitza pendulum oscillates around the equilibrium at π.

Liapunov stability

Theorem 1 *Let y_e be an equilibrium for $\dot{y} = F(y)$. Let $L : \mathcal{O} \to \mathbf{R}$ be differentiable, and let $\mathcal{O} \subset \mathbf{R}^n$ be an open set such that $y_e \in \mathcal{O}$.*

Suppose

(a) $L(y_e) = 0$, $L(y) > 0$ *for* $y \neq y_e$,

(b) $\frac{d}{dt}L \leq 0$ *in* $\mathcal{O} - \{y_e\}$,

then y_e is stable.