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Examination paper for **TMA4195 Mathematical Modeling**

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Problem 1 A physical system is governed by the following scaled initial value problem,

$$\frac{dy}{dt} = e^{-\epsilon y} \quad \text{for} \quad t > 0, \qquad y(0) = 1.$$

Use perturbation methods to find an approximation of the solution with $O(\epsilon^3)$ error.

Problem 2 A rectangular channel of length L and angle of incline θ leads surplus water away from a dam. The water level (the height) in the channel is H, and the main forces driving the flow is gravity and friction along the walls and bottom. Relevant physical quantities are mass density ρ ($\frac{\mathrm{kg}}{\mathrm{m}^3}$), gravitational acceleration g ($\frac{\mathrm{m}}{\mathrm{s}^2}$), wall/bottom roughness e (m), and viscosity μ ($\frac{\mathrm{kg}}{\mathrm{ms}}$).

A child drops a pinecone into the water and observes its motion as it floats down the channel. Based on all the physical quantities previously mentioned, find the most general dimensionally consistent model for the velocity U of the pinecone as it exits the channel.

Problem 3 A horizontal mass spring system with small cubic damping has the following governing equation and initial conditions:

$$m\frac{d^2x^*}{dt^{*2}} = -kx^* - r\left(\frac{dx^*}{dt^*}\right)^3 \quad \text{for} \quad t^* > 0, \qquad x^*(0) = x_0 > 0, \quad \frac{dx^*}{dt^*}(0) = 0.$$

where x^* is the position of the mass, with $x^* = 0$ when the mass is at rest and the spring being extended when $x^* > 0$ and compressed for $x^* < 0$. The quantities m, k, and r are the mass and the spring and damping constants respectively. The system is set in motion by moving the mass to position $x^* = x_0 > 0$ and then releasing it at time $t^* = 0$.

Find the natural space scale for this problem, and then determine the three natural time scales.

Scale the initial value problem when the first two terms in the equation (acceleration and spring force) dominate.

Problem 4 Let x^* be the size of the population of a remote fishing village and y^* the size of the population of fish in the sea nearby. To describe the evolution of x^* and y^*, a scaled model is proposed:

$$\begin{cases} \dfrac{dx}{dt} = -x + y, \\[2ex] 5\dfrac{dy}{dt} = y(1-y) - xy. \end{cases}$$

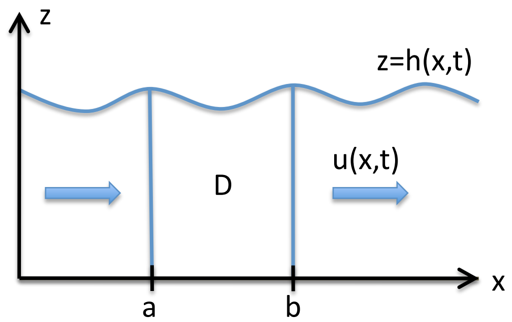
Give a physical interpretation of the different terms in this model.

Find the equilibrium points and determine their stability.

If $x(0), y(0) > 0$, what will $x(t)$ and $y(t)$ converge to as $t \to \infty$?

Problem 5 Part a), b), and c) can be solved independently of one another.

Water in a river of height h flows in positive x-direction with velocity u. We assume that the other components of the velocity are zero, that $u = u(x,t)$ and $h = h(x,t)$ (i.e. they are independent of (y,z)), and the water mass density ρ is constant. In the following we ignore the y-direction and consider a 2-dimensional model.



Conservation of mass can be expressed by the partial differential equation

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0. \tag{1}$$

a) Let D denote the part of the river which lies between the lines $x = a$ and $x = b$,

$$D = \{(x,z) : \ 0 \le z \le h(x,t), \ a \le x \le b \}.$$

Write down the equation for conservation of mass of water in the region D (the conservation law in integral form).

Assuming that u and h are smooth functions, use this equation to derive the partial differential equation (1).

We want to study flood waves in rivers under the simplifying assumption that gravity and friction forces are at equilibrium. This means that $u^2 = c^2 h$ for some constant c and equation (1) simplifies to

$$\frac{\partial h}{\partial t} + c \frac{\partial}{\partial x}(h^{\frac{3}{2}}) = 0. \tag{2}$$

We interpret shock solutions of this equation as flood waves.

b) Write down a shock solution of equation (2), including the correct shock speed.

Show that the shock solution h of (2) has to be a decreasing function of x:

$$h(a,t) \geq h(b,t) \qquad \text{for} \qquad a \leq b.$$

Hint: The method of characteristics can be useful for the second question.

Two of the largest rivers in western Europe, the north-westward moving Rhine and eastward moving Danube, regularly experience floods and flood waves. Floods in rivers are mainly caused by heavy rainfall, and in western Europe the wet weather systems predominantly move from the western oceans in an easterly direction.

c) According to some people, there are more flood waves in the Danube than in the Rhine.

Explain this using our model. Show any claim you make about the model.

Problem 6

The country road from town A to town B has speed limit 60 km/h, except where it passes through village C and the speed limit is 30 km/h. To study the effect on traffic of the speed reduction, engineers have developed a scaled model for traffic in one of the directions:

$$
\begin{aligned}
&\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} j(\rho) = 0, && x \in (-\infty, -1) \cup (1, \infty), && t > 0, && \text{[outside the village]} \\
&\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} \frac{1}{2} j(\rho) = 0, && x \in (-1, 1), && t > 0, && \text{[in the village]} \\
&\rho(x, 0) = \begin{cases} \frac{1}{2}, & x \in (-\infty, 1), \\ 0, & x \in (1, \infty), \end{cases} && && t = 0, && \text{[initial condition]} \\
&\frac{1}{2} j(\rho(1^-, t)) = j(\rho(1^+, t)), && x = 1, && t > 0, && \text{[boundary condition]} \\
&j(\rho(-1^-, t)) = \frac{1}{2} j(\rho(-1^+, t)), && x = -1, && t > 0, && \text{[boundary condition].}
\end{aligned}
$$

In this model, the village occupy the part $-1 \leq x \leq 1$ of the road, ρ is the scaled car density, and $j(\rho) = \rho(1 - \rho)$ and $\frac{1}{2} j(\rho)$ are the scaled car fluxes outside and inside the village respectively.

Assuming the car flux is always maximal in the village, determine the solution $\rho(x, t)$ for all $x > 1$ and $t > 0$.