Norwegian University of Science and Technology
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Exam in TMA4195 Mathematical Modeling 14.12.2016 Solutions

Problem 1 We insert

$$y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \mathcal{O}(\epsilon^3)$$

into

$$\dot{y} = e^{-\epsilon y}, \qquad y(0) = 1$$

and expand the exponential to find that

$$\begin{aligned}\dot{y}_0 + \epsilon\dot{y}_1 + \epsilon^2\dot{y}_2 + \mathcal{O}(\epsilon^3) &= 1 - \epsilon y + \frac{1}{2}\epsilon^2 y^2 + \mathcal{O}(\epsilon^3)\\ &= 1 - \epsilon(y_0 + \epsilon y_1 + \mathcal{O}(\epsilon^2)) + \frac{1}{2}\epsilon^2(y_0 + \mathcal{O}(\epsilon))^2 + \mathcal{O}(\epsilon^3)\\ &= 1 - \epsilon(y_0 + \epsilon y_1) + \frac{1}{2}\epsilon^2 y_0^2 + \mathcal{O}(\epsilon^3)\end{aligned}$$

and

$$y_0(0) + \epsilon y_1(0) + \epsilon^2 y_2(0) + \mathcal{O}(\epsilon^3) = 1 + 0\cdot\epsilon + 0\cdot\epsilon^2 + \mathcal{O}(\epsilon^3).$$

Since these equalities hold for all ϵ, it follows that

$$\begin{aligned}\dot{y}_0 &= 1, & y_0(0) &= 1,\\ \dot{y}_1 &= -y_0, & y_1(0) &= 0,\\ \dot{y}_2 &= -y_1 + \frac{1}{2}y_0^2, & y_2(0) &= 0.\end{aligned}$$

Solving first for y_0, we get $y_0 = t + 1$, which we then insert into the equation for y_1 to get

$$\dot{y}_1 = -(t+1), \qquad y_1(0) = 0 \qquad \Rightarrow \qquad y_1 = \int_0^t (-\tau - 1)\,d\tau = -\frac{1}{2}t^2 - t.$$

Inserting this into the equation for y_2 we get

$$\dot{y}_2 = \frac{1}{2}t^2 + t + \frac{1}{2}(t+1)^2 = t^2 + 2t + \frac{1}{2}, \qquad y_2(0) = 0 \qquad \Rightarrow \qquad y_2 = \frac{1}{3}t^3 + t^2 + \frac{1}{2}t.$$

In total,

$$y(t) = t + 1 - \epsilon(\frac{1}{2}t^2 + t) + \epsilon^2(\frac{1}{3}t^3 + t^2 + \frac{1}{2}t) + \mathcal{O}(\epsilon^3).$$

Problem 2 We may assume a relation

$$\Phi(U, L, H, \theta, \rho, g, e, \mu) = 0. \tag{1}$$

The dimensional matrix A is

	L	H	θ	ρ	g	e	μ	U
m	1	1	0	-3	1	1	-1	1
s	0	0	0	0	-2	0	-1	-1
kg	0	0	0	1	0	0	1	0

The rank of A is 3 and we thus have $8 - 3 = 5$ dimensionless combinations. Natural choices for core variables (do not choose U!) are L, ρ and g. The dimensionless combinations are then

$$\pi_1 = \frac{U}{L^\bullet \rho^\bullet g^\bullet} = \frac{U}{L^{\frac{1}{2}} g^{\frac{1}{2}}}, \qquad \pi_2 = \theta, \qquad \pi_3 = \frac{e}{L}, \qquad \pi_4 = \frac{H}{L},$$

$$\pi_5 = \frac{\mu}{L^\bullet \rho^\bullet g^\bullet} = \frac{\mu}{\rho g^{\frac{1}{2}} L^{\frac{3}{2}}} \qquad \text{or} \qquad \pi_5' = \pi_5^2 = \frac{\mu^2}{\rho^2 g L^3}.$$

The second part of Buckingham's Pi Theorem states that any physical relation (1) is equivalent to a relation between the associated dimensionless combinations

$$\Psi(\pi_1, \pi_2, \pi_3, \pi_4, \pi_5) = 0.$$

Solving for π_1 would give the relation

$$\pi_1 = \bar{\Psi}(\pi_2, \pi_3, \pi_4, \pi_5),$$

or

$$U = L^{\frac{1}{2}} g^{\frac{1}{2}} \bar{\Psi}(\theta, \frac{e}{L}, \frac{H}{L}, \frac{\mu}{\rho g^{\frac{1}{2}} L^{\frac{3}{2}}}).$$

This is the most general dimensionally consistent model for U.

Problem 3 Because of the damping,

$$\max |x^*(t)| \leq x_0,$$

and hence the natural space scale is x_0.

Time scales are found by balancing terms in the equation, setting

$$x^* = x_0 x, \qquad t^* = Tt, \quad \text{where } t, x, x', x'' \sim \mathcal{O}(1).$$

By the equation,

$$m\frac{x_0}{T^2}\frac{d^2x}{dt^2} = -kx_0x - r\left(\frac{x_0}{T}\frac{dx}{dt}\right)^3.$$

Balancing the first and second term:

$$\frac{m}{T^2} \sim k \quad \Rightarrow \quad T \sim \sqrt{\frac{m}{k}}$$

Balancing the first and third term:

$$\frac{m}{T^2} \sim \frac{rx_0^2}{T^3} \quad \Rightarrow \quad T \sim \frac{rx_0^2}{m}$$

Balancing the second and third term:

$$k \sim \frac{rx_0^2}{T^3} \quad \Rightarrow \quad T \sim \left(\frac{rx_0^2}{k}\right)^{\frac{1}{3}}$$

The three natural time scales are: $\sqrt{\frac{m}{k}}$, $\frac{rx_0^2}{m}$ and $\left(\frac{rx_0^2}{k}\right)^{\frac{1}{3}}$.

When the first two terms dominate, $T = \sqrt{\frac{m}{k}}$. The equation then becomes

$$kx_0\frac{d^2x}{dt^2} = -kx_0x - r\frac{x_0^3}{\left(\frac{m}{k}\right)^{\frac{3}{2}}}\left(\frac{dx}{dt}\right)^3,$$

or

$$\frac{d^2x}{dt^2} = -x - \epsilon\left(\frac{dx}{dt}\right)^3, \qquad \epsilon = r\frac{x_0^2k^{\frac{1}{2}}}{m^{\frac{3}{2}}}.$$

Problem 4

Humans: Population depends on the amount of fish in the ocean. More fish means more money/work, and thus more people. Without the fish, people move away.

Fish: The model is a logistic growth model plus a death/harvest term ($-xy$). The harvest rate depends on the size of the human population. With no humans, the fish population would follow logistic growth ($y(1-y)$ after re-scaling) and converge to the carrying capacity $y = 1$.

The equilibrium points are the solution of $F(x, y) = 0$, where

$$F(x,y) = \begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}-x+y\\ \frac{1}{5}(y(1-y)-xy)\end{pmatrix}.$$

Inserting $x = y$ into the equation for $\dot{y}$, we get

$$0 = y - y^2 - xy = y(1 - 2y),$$

which we solve to get the two equilibrium points, $(0, 0)$ and $(\frac{1}{2}, \frac{1}{2})$.

To determine the stability we compute the Jacobi matrix of the right hand side F:

$$DF(x, y) = \begin{pmatrix} -1 & 1 \\ -\frac{1}{5}y & \frac{1}{5}(1 - 2y - x) \end{pmatrix}$$

At the equilibrium point $(0, 0)$,

$$DF(0, 0) = \begin{pmatrix} -1 & 1 \\ 0 & \frac{1}{5} \end{pmatrix},$$

with eigenvalues $\lambda_+ = \frac{1}{5}$ and $\lambda_- = -1$. Since $\max_\pm \mathrm{Re}(\lambda_\pm) > 0$ (one eigenvalue is positive), we can conclude that $(0, 0)$ is unstable (here it is a saddle point).

At the equilibrium point $(\frac{1}{2}, \frac{1}{2})$,

$$DF\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \begin{pmatrix} -1 & 1 \\ -\frac{1}{10} & -\frac{1}{10} \end{pmatrix}$$

We solve to find the eigenvalues of this matrix,

$$\det\left(DF\left(\tfrac{1}{2}, \tfrac{1}{2}\right) - \lambda I\right) = 0 \quad \Rightarrow \quad \lambda^2 + \frac{11}{10}\lambda + \frac{2}{10} = 0 \quad \Rightarrow \quad \lambda_\pm = -\frac{11}{20} \pm \frac{\sqrt{41}}{20}.$$

Since $\max_\pm \mathrm{Re}(\lambda_\pm) < 0$ (both are negative), $(\frac{1}{2}, \frac{1}{2})$ is a stable equilibrium point (a stable node here).

From the stability analysis of the two equilibrium points, we can conclude that for $x(0), y(0) > 0$, $(x(t), y(t))$ will converge towards the stable equilibrium point $(\frac{1}{2}, \frac{1}{2})$ as $t \to \infty$.

Problem 5

a) Conservation of mass in D:

$$\frac{d}{dt}\iint_D \rho\, dx\, dz = -\int_{\partial D} \vec{j} \cdot \vec{n}\, dS + 0$$
$$\text{change in } D = \text{flux in/out} + \text{production}$$

where the flux is $\vec{j} = \rho u(x,t) \cdot \vec{e}_x$, and $\vec{n}$ is the outward normal vector at the boundary ∂D of D. There is no flux across the top and bottom part of the boundary ∂D, so

$$\int_{\partial D} \vec{j} \cdot \vec{n} \, dS = \int_{\partial D \cap \{x=a\}} \vec{j} \cdot \vec{n} \, dS + \int_{\partial D \cap \{x=b\}} \vec{j} \cdot \vec{n} \, dS.$$

Since $\vec{n} = -\vec{e}_x$ at $x = a$ and $\vec{n} = \vec{e}_x$ at $x = b$,

$$\int_{\partial D \cap \{x=a\}} \vec{j} \cdot \vec{n} \, dS = \int_0^{h(a,t)} \rho u(a,t)(-1) \, dz = -\rho (hu)(a,t),$$
$$\int_{\partial D \cap \{x=b\}} \vec{j} \cdot \vec{n} \, dS = \int_0^{h(b,t)} \rho u(b,t)(+1) \, dz = +\rho (hu)(b,t).$$

Moreover,

$$\iint_D \rho \, dx \, dz = \int_a^b \int_0^{h(x,t)} \rho \, dx \, dz = \rho \int_a^b h(x,t) \, dx.$$

Hence, the conservation of mass in D equation takes the form

$$\frac{d}{dt} \int_a^b h(x,t) \, dx = -(hu)(b,t) + (hu)(a,t).$$

To find the PDE, take $a = x$ and $b = x + \Delta x$, divide by Δx, and interchange $\frac{d}{dt}$ and $\int \dots dx$;

$$\frac{1}{\Delta x} \int_x^{x+\Delta x} h_t(y,t) \, dy = -\frac{(uh)(x + \Delta x, t) - (uh)(x,t)}{\Delta x}.$$

Then we send $\Delta x \to 0$, noting that

$$\lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_x^{x+\Delta x} h_t(y,t) \, dy = \lim_{\Delta x \to 0} \Big(h_t(x,t) + \max_{|x-y|<\Delta x} \big| h_t(y,t) - h_t(x,t) \big| \Big) = h_t(x,t)$$

since h_t is continuous, and find that

$$h_t(x,t) = -\frac{\partial}{\partial x} (uh)(x,t)$$

for any point x and $t > 0$.

b) A shock solution starting at $x = 0$ is given by

$$h(x,t) = \begin{cases} h_l, & x < s(t), \\ h_r, & x > s(t), \end{cases}$$

where $h_l \neq h_r$ are constants and the shock curve $s(t)$ satisfies the Rankine–Hugoniot condition

$$\dot{s} = \frac{j(h_l) - j(h_r)}{h_l - h_r} = c\frac{h_l^{\frac{3}{2}} - h_r^{\frac{3}{2}}}{h_l - h_r}, \qquad s(0) = 0,$$

i.e. $s(t) = \dot{s} \cdot t$.

To check whether $h(x,t)$ is increasing or decreasing, we note that h is the unique solution of the Riemann problem

$$\begin{cases} h_t + c\frac{\partial}{\partial x} h^{\frac{3}{2}} = 0, \\ h(x,0) = \begin{cases} h_l, & x < 0, \\ h_r, & x > 0. \end{cases} \end{cases}$$

This problem has a shock solution if and only if the characteristics collide.

The characteristic equations are $(z(t) = h(x(t),t))$

$$\begin{cases} \dot{x} = j'(z) = \frac{3}{2} c z^{\frac{1}{2}} := c(z), & x(0) = x_0 \\ \dot{z} = 0, & z(0) = h(x(0),0) = h(x_0,0) = \begin{cases} h_l, & x_0 < 0, \\ h_r, & x_0 > 0, \end{cases} \end{cases}$$

where the function $c(z) = j'(z)$ is different from the constant c. The solutions are

$$x(t) = x_0 + c(h(x_0,0))t.$$

We get collisions if and only if the left characteristic overtake right characteristic, i.e. if

$$c(h_l) > c(h_r), \qquad \text{or equivalently,} \qquad h_l > h_r.$$

Hence the shock solution $h(x,t)$ is decreasing in x.

c) When rainfall moves from west to east, water levels/height will increase in the western part of the river first. The level h of the rivers is thus decreasing from west to east. Since this is the direction of flow of the Danube, and against the flow direction in the Rhine, shocks/flood waves will form in the first river but not in the second.

The reason for this is that shocks form if and only if $(h_0)_x < 0$, i.e. h_0 is decreasing somewhere. To show this, we use the *first shock* analysis from the notes of Krogstad: we find the first time two nearby characteristic curves $x(t)$ and $y(t)$ collide. If $x(t)$ and $y(t)$ start at x_0 and $x_0 + \Delta x$ respectively, then from part b),

$$\begin{aligned} x(t) &= x_0 + t \cdot c(h(x_0)), \\ y(t) &= x_0 + \Delta x + t \cdot c(h(x_0 + \Delta x)). \end{aligned}$$

We have collisions at time t if $x(t) = y(t)$, which means

$$\Delta x = t\Big(- c\,(h\,(x_0 + \Delta x)) + c\,(h(x_0))\Big) \qquad \text{or} \qquad t = \frac{-1}{\frac{c(h(x_0+\Delta x)) - c(h(x_0))}{\Delta x}}.$$

Since nearby characteristics collide first, the first time $x(t)$ collides with another characteristic is

$$t_c(x_0) = \lim_{\Delta x \to 0} \frac{-1}{\frac{c(h(x_0+\delta x)) - c(h(x_0))}{\Delta x}} = -\frac{1}{\frac{d}{dx}\left[c(h_0(x)\right]_{x=x_0}}.$$

Since

$$\frac{d}{dx} c(h_0(x)) = c'(h_0(x)) \cdot {h_0}'(x) = c\frac{3}{4}(h_0(x))^{-\frac{1}{2}} \cdot {h_0}'(x),$$

we have that

$$t_c(x_0) > 0 \qquad \text{if and only if} \qquad {h_0}'(x_0) < 0.$$

Hence, we will get a shock (in positive time) if and only if there exists an x_0 such that ${h_0}'(x_0) < 0$.

Problem 6

We let $j_v(\rho) := \frac{1}{2}j(\rho)$ be scaled flux inside the village ($-1 \leq x \leq 1$), and find its maximum:

$$\frac{d}{d\rho} j_v(\rho) = 0 \quad \Rightarrow \quad \frac{1}{2}(1 - 2\rho) = 0 \quad \Rightarrow \quad \rho = \frac{1}{2},$$

and

$$\max_{\rho} j_v(\rho) = j_v(\frac{1}{2}) = \frac{1}{2}\frac{1}{2}(1 - \frac{1}{2}) = \frac{1}{8}.$$

Hence by the flux condition at $x = 1$,

$$j(\rho) = j_v(\rho) = \frac{1}{8} \quad \text{at} \quad x = 1.$$

We convert this to a boundary condition on ρ:

$$j(\rho) = \frac{1}{8} \quad \Leftrightarrow \quad \rho(1-\rho) = \frac{1}{8} \quad \Leftrightarrow \quad \rho^2 - \rho + \frac{1}{8} = 0 \quad \Leftrightarrow \quad \rho^{\pm} = \frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{1}{2}},$$

Only inflow conditions can be imposed, and in the domain $x > 1$ inflow means positive characteristics speed $c(\rho)$. Here $c(\rho) = j'(\rho) = 1 - 2\rho$ and

$$c(\rho^{\pm}) = 1 - 2\rho^{\pm} = \mp\frac{\sqrt{2}}{2}.$$

Hence $c > 0$ for $\rho = \rho^- = \frac{1}{2}\left(1 - \frac{\sqrt{2}}{2}\right)$.

The correct boundary value problem then becomes

$$\begin{cases} \rho_t + \frac{\partial}{\partial x} j(\rho) = 0, & x > 1, t > 0, \\ \rho = \rho^-, & x = 1, t > 0, \\ \rho = 0, & x > 1, t = 0. \end{cases}$$

This we solve by the method of characteristics ($z(t) = \rho(x(t), t)$):

$$\begin{cases} \dot{x} = j'(z) = c(z), & x(t_0) = x_0, \\ z = 0, & z(t_0) = \rho(x(t_0), t_0). \end{cases}$$

The solution is

$$x(t) = x_0 + (t - t_0) \cdot c(\rho(x(t_0), t_0).$$

Characteristics from $t = 0$ ($t_0 = 0$, $\rho(x(t_0), t_0) = \rho_0(x_0) = 0$):

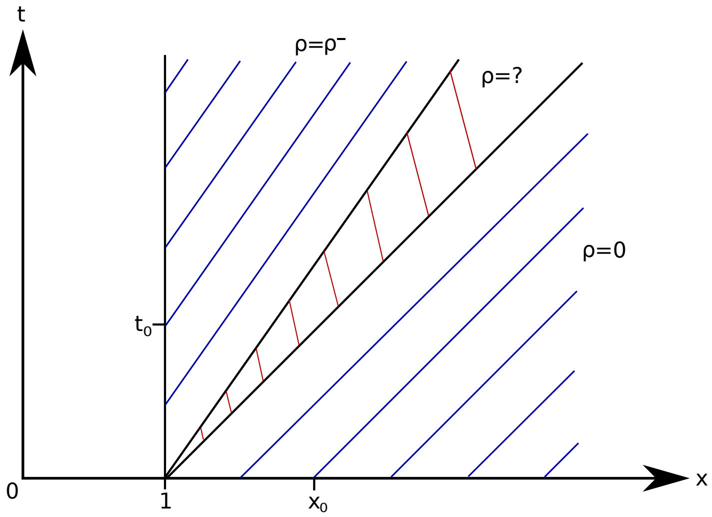
$$x(t) = x_0 + t \cdot c(0) = x_0 + t.$$

Characteristics from the boundary ($x_0 = 1$, $\rho(x_0, t) = \rho^-$):

$$x(t) = x_0 + t \cdot c(\rho^-) = x_0 + \frac{\sqrt{2}}{2} t.$$

Since $c(0) = 1 > \frac{\sqrt{2}}{2} = c(\rho^-)$, the characteristics from the boundary and the initial line will move away from each other and there will be a dead sector starting at $x = 1$, $t = 0$:

$$\frac{\sqrt{2}}{2} t \leq x - 1 \leq t.$$



Hence, the physical solution in the sector is a rarefaction wave, i.e. a solution of the form

$$\rho(x,t) = \varphi\left(\frac{x-1}{t}\right).$$

Inserted into the equation, we get

$$0 = \rho_t + \frac{\partial}{\partial x} j(\rho) = -\varphi' \cdot \frac{x-1}{t^2} + c(\varphi)\varphi' \frac{1}{t},$$

and multiplying by $\frac{t}{\varphi'}$ (assuming $\varphi' \neq 0$),

$$c(\varphi) = \frac{x-1}{t} \quad \Rightarrow \quad 1 - 2\varphi = \frac{x-1}{t} \quad \Rightarrow \quad \varphi = \frac{1}{2}\left(1 - \frac{x-1}{t}\right).$$

The total solution is then

$$\rho(x,t) = \begin{cases} 0, & x-1 > t, \\ \frac{1}{2}\left(1 - \frac{x-1}{t}\right), & \frac{\sqrt{2}}{2} t < x-1 < t, \\ \rho^-, & 0 < x-1 < \frac{\sqrt{2}}{2} t. \end{cases}$$