TMA4205 - Autumn 2012

Assignment 2

Problem 1.

(a) We shall study some properties of symmetric positive definite (SPD) matrices Suppose $A \in \mathbb{R}^{n\times n}$ is symmetric. Then A is SPD if

(i) $x^T Ax > 0$, for all $x \in \mathbb{R}^n$, $x \neq 0$

(ii) all eigen values of A are positive

Show that these two conditions are equivalent

(b) Show the following equivalence

$$A \text{ is SPD} \quad \Leftrightarrow \quad A^{-1} \text{ is SPD}$$

(c) Let $A \in \mathbb{R}^{n\times n}$ and let $x \in \mathbb{R}^n$. The expression

$$R(x) = \frac{x^T Ax}{x^T x}$$

is called the Rayleigh quotient of A. If A is SPD, show that

$$\lambda_1 \leq R(x) \leq \lambda_n, \quad \forall x \in \mathbb{R}^n,\ x \neq 0$$

where λ_1 and λ_n are the smallest and largest eigenvalue of A respectively.

Problem 2. Consider the matrix

$$A = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$$

(a) Is A normal?

(b) Find the eigenvalues and eigenvectors of A

(c) Are the eigenvectors linearly independent? Are they orthogonal?

(d) Can A be diagonalized?

(e) Show that $u^T Au \geq \alpha \|u\|_2^2$ for all $u \in \mathbb{R}^2$. What is the largest possible value for α?

(f) Is A positive definite?

(g) Does A have a Cholesky factorization?

(h) Find a Schur factorization for A.

(i) Find $\exp(A)$.

Problem 3. We consider again the Poisson problem

$$\begin{aligned} -\Delta u &= f & \text{in } \Omega = (0,1)\times(0,1) \\ u &= 0 & \text{on } \partial\Omega \end{aligned}$$

We discretize the system on a uniform grid with stepsize $h = \frac{1}{n}$ in each direction.

(a) Solve the resulting system of linear equation in MATLAB with three different methods

- (i) The diagonalization method discussed in the lectures
- (ii) LU-factorization with sparse matrices
- (iii) Full LU-faktorisering without sparse matrices.

(b) Does the timings scale as expected?

(c) Calculate the condition number of the discrete Laplacian for various different values of n. How does the condition number $\kappa(A)$ scale with n?

(d) How does the condition number $\kappa(A)$ scale for the one-dimensional Laplacian? Compare these results with the results from questions (c).