TMA4205 - Autumn 2012

Assignment 5

Problem 1. (From Saad, Chap 5) In Chapter 5.3.3 in the book it shown that using a one dimensional projection method with $\mathcal{K} = \text{span}\{A^T r\}$ and $\mathcal{L} = \text{span}\{AA^T r\}$ is equivalent to using the Steepest Descent Method on the normal equations $A^T Ax = A^T b$.

Show that an *orthogonal* projection method for $A^T Ax = A^T b$ with approximation space $\mathcal{K}$ is equivalent to applying a projection method to $\mathcal{K}$ orthogonal to $\mathcal{L} = A\mathcal{K}$ for the problem $Ax = b$.

Problem 2. Algorithm 6.1 in Saad is implemented in the attached MATLAB -function **arnoldi_gs.m**. This algorithm constructs an orthogonal basis for the Krylov subspace $\mathcal{K}_m(A, v)$ based on a classical Gram-Schmidt procedure. Test this function on the matrix A generated by the attached **poisson2.m** for different values of m and $N = n^2$. For instance, choose $N = 100$, $v = e_1$ and $m = 10, 20, 30, 40, 50$.

(a) Test to what extent the relation between A, V_m and H_m from Proposition 6.5 in Saad is fulfilled. Also check if the vectors $v_1, \ldots, v_m$ really are orthonormal, i.e. check whether $V_m^T V_m = I$ (exactly).

(b) Modify the function **arnoldi_gs.m** such that it uses modified Gram-Schmidt for. Repeat the experiments from the previous question.

Problem 3.

(a) If A is symmetric and positive definite (SPD), show that A^{-1} also is SPD and can be used to define a norm on $\mathbb{R}^n$ via
$$\|v\|_{A^{-1}} = \left(v^T A^{-1} v\right)^{1/2}$$

(b) We know that the Conjugate Gradient Method will minimize the error in A-norm over all elements in the Krylov subspace $\mathcal{K}_m(A, r_0)$. Show that the algorithm also, in each iteration, will minimize the associated residual in A^{-1}-norm.

(c) Each update of the solution in the Conjugate Gradient Method can be expressed as $x_{j+1} = x_j + \alpha_j\, p_j$ where $\alpha_j = (r_j, r_j)/(Ap_j, p_j)$; see Algorithm 6.18 in Saad. Show that α_j is optimal in the sense that it minimizes the functional $f(w) = \frac{1}{2} w^T Aw - w^T b$, $f : \mathbb{R}^n \to \mathbb{R}$, along the search direction p_j.

Problem 4 Suppose we want to use conjugate gradient(CG) method to solve the linear system $Ax = b$ where A is SPD, and we are given a SPD preconditioner M. Using left- or right-preconditioning would lead to a preconditioned system with matrix $M^{-1}A$ or AM^{-1}. We know that the resulting matrix is no longer symmetric. Therefore in order to adapt Algorithm 6.18 in Saad, we must re-define the inner-products so that the matrix of the preconditioned system is self-adjoint with respect to the new inner-product.

(a) Show that in either the left- or right-preconditioned system the preconditioned matrices $M^{-1}A$ and AM^{-1} are positive-definite.

(b) Show that there exists a SPD matrix N such that $M = N^2$.
[*Hint*: M symmetric $\Longrightarrow$ M is unitarily diagonalizable.]

(c) Using $M = N^2$ as a split-preconditioner, show that the resulting preconditioned CG (PCG) algorithm is equivalent to Algorithm 9.1 in [Saad, p.263].

(d) How do the condition numbers of the preconditioned matrices $N^{-1}AN^{-1}$, $M^{-1}A$ and AM^{-1} compare?

(e) From the estimate of convergence of the CG algorithm, can you deduce an estimate for the convergence of the PCG algorithm?