TMA4205 - Autumn 2012. Assignment 6

Obligatory assignment: Counts 20% of the final grade

Due date: Friday, November 23, 2012

Instructions: Solutions to the assignment must be typed and submitted in **pdf** files via email. You can work in groups of at most 2 persons, and remember to write your **student ID** number. Your report should include all figures, tables and MATLAB codes used.

Problem 1. (From Saad, Chap.13, Exercise 10) This exercise describes the the two-level multgrid V-cycle for the 1D Poisson problem discretized via finite differences. The number of pre- and post-smoothing iterations is chosen as $\nu_1 = \nu_2 = 2$ or $\nu_1 = \nu_2 = 5$.

(a) Repeat this exercise using each of the following iterations:

  (i) Gauss-Seidel;

  (ii) Red-black Gauss-Seidel (See Briggs *et al.* Chap. 2);

  (iii) Weighted Jacobi with weight parameter ($\omega = \frac{2}{3}$).

(b) Which of these iterations give optimal results?

Problem 2.

(a) Exercise 2, Saad, Chap.9, p.279. *Hint:* The algorithm might require 1 or more extra vectors for storage.

(b) Estimate the number of iterations for the PCG algorithm to converge. *Hint:* Estimate the number of iterations required in order to reduce the initial error with 10 orders of magnitude; express your answer in terms of the condition number of the preconditioned system.

(c) Consider the overlapping additive Schwarz preconditioner discussed on page 6, of the lecture notes. Prove that the preconditioner M^{-1} is symmetric and positive definite.

(d) Consider now the symmetrized version of the overlapping multiplicative Schwarz preconditioner; see page 6. Again, prove that the preconditioner M^{-1} is symmetric and positive definite. *Hint:* Find an explicit expression for M^{-1}.

Problem 3. Consider the two dimensional Poisson problem

$$-\Delta U = f, \text{ in } \Omega = (-1,1) \times (0,1) \tag{1}$$

$$U = 0, \text{ on } \partial\Omega, \tag{2}$$

where $\Delta U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}$, and $f(x,y) = \pi^2(1 - 5\cos 2\pi y)\sin \pi x, \quad (x,y) \in \Omega$. We discretize this problem using the 5-point finite difference method on a uniform grid denoted by $x_i = -1 + ih$, $y_j = jh$ with $i = 0, \dots, 2N$, $j = 0, \dots, N$ and $h = 1/N$. The discrete system of equations can be expressed as $Au = b$, where A is the discrete Laplace operator, u is the unknown vector, and b is the known right-hand side. In the implementations, the matrix A must not be explicitly constructed, but a function that describes the action of A on a vector.

(a) Solve the system $Au = b$ using the preconditioned conjugate gradient method. You should implement your own version of this algorithm. Set the preconditioner to be the identity operator, i.e., consider first the unpreconditioned case. Plot $\log_{10}(\|r\|_2)$ as a function of the iteration number, where r is the residual vector. Are your results consistent with what you would expect from Problem 2(b)?

(b) Use the diagonal of A as a preconditioner. Does this help the convergence rate? Explain your findings.

The domain is decomposed into two equal halves Ω_1 an Ω_2 as shown in Figure 1, and the overlapping domains $\widetilde{\Omega}_1$ and $\widetilde{\Omega}_2$ overlap by an amount $\delta = h$.

(c) Use the additive Schwarz preconditioner discussed in Problem 2(c). Plot the convergence behavior and discuss your findings.

(d) Use the multiplicative Schwarz preconditioner discussed in Problem 2(d). Plot the convergence behavior and discuss your findings.

(e) Compare the number of iterations used in (d) and (e).

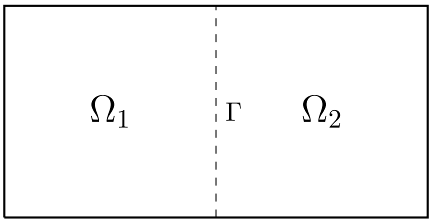

Figure 1: Domain Ω decomposed into two equal subdomains Ω_1 and Ω_2. The overlapping subdomains $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ overlap at Γ with an overlap amount $\delta = h$.