Exam in Numerical Linear Algebra (TMA4205)

Monday, December 17, 2012
Time: 09:00 - 13:00
Grades available: January 7, 2013

Aids: Code C - The following printed/hand written aids are allowed.

- Y. Saad, Iterative Methods for Sparse Linear Systems, 2nd ed.
- Trefethen and Bau, Numerical linear algebra or Notes from the same book
- Golub and Van Loan, Matrix Computations or Notes from the same book
- Own lecture notes from the course including handouts from course.

Problem 1 The real *nonsingular* matrix

$$A = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 3 & -1 & 1 \\ 0 & -1 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix}$$

is unitarily similar to an upper Hessenberg matrix H.

a) Show that H is also tridiagonal.

b) Is H positive-definite?
[*Hint*: Use Gerschgorin's theorem to investigate bounds for the eigenvalues].

c) Use Householder reflections to find a unitary matrix Q so that $H = Q^T A Q$.
[You may use the information given at the footnote on the last page].

Problem 2

a) Consider a matrix of the form

$$A = I + \mu S,$$

where μ is a scalar and S is skew-symmetric, and I is the identity matrix.

i) Show that A is positive definite.

ii) Consider the Arnoldi process for A. Show that the resulting Hessenberg matrix will have the following tridiagonal form:

$$H_m = \begin{bmatrix} 1 & -\eta_2 & & & \\ \eta_2 & 1 & -\eta_3 & & \\ & & \cdots & & \\ & & \eta_{m-1} & 1 & -\eta_m \\ & & & \eta_m & 1 \end{bmatrix}.$$

b) Suppose $Ax = b$ is a linear system where A is symmetric and positive-definite. Consider an orthogonal projection method for which the search and constraint spaces are given by $\mathcal{L} = \mathcal{K} = \operatorname{span}\{r_0, Ar_0\}$ where $r_0 = b - Ax_0$ is the current residual. Let $\tilde{x} = x_0 + \alpha r_0 + \beta A r_0$ denote the solution update, where α and β are some real constants. Prove the error satisfies:

$$\|\tilde{e}\|_A^2 = \left[1 - \alpha\frac{(r_0, r_0)}{(A^{-1}r_0, r_0)} - \beta\frac{(Ar_0, r_0)}{(A^{-1}r_0, r_0)}\right] \|e_0\|_A^2$$

where $\tilde{e} = x - \tilde{x}$, $e_0 = x - x_0$, $\|y\|_A = \sqrt{(Ay, y)}$. Derive the expressions for α and β. Find an expression for the lower bound of the term $\beta\dfrac{(Ar_0, r_0)}{(A^{-1}r_0, r_0)}$. The answer must be expressed only in term of $\lambda_{\min}$ and $\lambda_{\max}$, the minimum and maximum eigenvalues of A respectively. [Hint: The spectrum of an SPD matrix provides a bound for its Rayleigh quotients].

Problem 3 In a finite difference discretization of the one-dimensional Poisson equation

$$\begin{aligned} -u_{xx} &= f, \text{ in } \Omega = (0, 1), \\ u(0) &= u(1) = 0, \end{aligned} \tag{1}$$

on a uniform grid we obtain a linear system $Ax = b$ where $A = \operatorname{tridiag}(-1, 2, -1) \in \mathbb{R}^{(n-1)\times(n-1)}$, $x \in \mathbb{R}^{n-1}$ is the vector of unknowns, and $h = \frac{1}{n}$ is the mesh parameter. The eigenvalues of A are known to be given by

$$\lambda_m = 2 - 2\cos\left(\frac{m\pi}{n}\right), \quad m = 1, \ldots, n-1.$$

The symmetric successive overrelaxation (SSOR) method for solving the system $Ax = b$ is an iterative method that can be described in the following two steps

$$\begin{aligned} (D - \omega E)x_{k+1/2} &= [\omega F + (1-\omega)D)]\, x_k + \omega b, \\ (D - \omega F)x_{k+1} &= [\omega E + (1-\omega)D)]\, x_{k+1/2} + \omega b, \end{aligned}$$

$k = 0, 1, \ldots$, where ω is the relaxation parameter, and D is the diagonal part of A, while E and F define, respectively, the lower and the upper triangular parts of A so that $A = D - E - F$. The iteration matrix is defined as a matrix G so that the iterative method can be expressed as a fixed point iteration $x_{k+1} = Gx_k + f$.

a) Prove that the iteration matrix for the SSOR method can be expressed as

$$G_\omega = I - \omega(2-\omega)(D - \omega F)^{-1} D (D - \omega E)^{-1} A,$$

where I is the identity matrix.

b) To use the SSOR as preconditioning for the conjugate gradient method we consider a splitting of the form $A = M - N$, so that the SSOR iteration takes the form

$$x_{k+1} = M^{-1}Nx_k + M^{-1}b = x_k + M^{-1}r_k,$$

where r_k is the residual vector, and $G = M^{-1}N$ is the iteration matrix. The matrix M is the required preconditioner. Assume that $\omega \in (0,2)$. Deduce from G_ω an expression for the preconditioner $M = M_\omega$, and show that it is symmetric and positive-definite. [*Hint*: In this example $F = E^T$].

c) For $\omega \in (0,2)$ the matrix $M_\omega^{-1}A$ of the preconditioned system is known to have condition number $\dfrac{2an^2 + \lambda_{\min}}{(2-\omega)\lambda_{\min}}$, where $a = \dfrac{(2-\omega)^2}{4\omega}$ and $\lambda_{\min}$ is the minimum eigenvalue of A. Show that the optimal value of ω for the preconditioner M_ω is given by

$$\omega_{\text{opt}} = \frac{2\sqrt{\gamma}}{1+\sqrt{\gamma}},$$

where $\gamma = \dfrac{n^2}{\lambda_{\min}}$. Assume that $n \to \infty$. How does the condition number of $M_\omega^{-1}A$ varies as a function of n when $\omega = \omega_{\text{opt}}$?

Problem 4 Consider the matrix

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}.$$

a) Under which conditions can the normal equation $A^TAx = A^Tb$ be solved using conjugate gradient iterations? Are these conditions satisfied in this case?

b) Find the condition number of A^TA in the 2-norm. Would you expect any improvement in the conjugate gradient method by using diagonal preconditioning?

c) Calculate the singular values of A.

d) Find the 1-, 2-, ∞- and Frobenius-norms of A?

Some useful information:

For problem 1. In the Householder reduction into Hessenberg form, the first Householder reflector applied to the left and right of A, reduces it to the matrix

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 1 & 1 & 3 \end{bmatrix}.$$

Kantorovich inequality. Suppose $B \in \mathbb{R}^{n\times n}$ is a symmetric and positive definite matrix. Then

$$\frac{(Bx,x)(B^{-1}x,x)}{(x,x)^2} \leq \frac{(\lambda_{\max}+\lambda_{\min})^2}{4\lambda_{\max}\lambda_{\min}},$$

for all $x \in \mathbb{R}^n,\ x \neq 0$, where $\lambda_{\min}$ and $\lambda_{\max}$ represent the smallest and largest eigenvalues of B respectively.

Condition number. The condition number of a matrix A is given by the formula $\kappa(A) = \|A\|\|A^{-1}\|$. In the p-norm, this is given by $\kappa_p(A) = \|A\|_p\|A^{-1}\|_p$.