Eigenvalues of tridiagonal Toeplitz matrices

Suppose we have an $n \times n$-matrix of the form

$$A = \begin{bmatrix} \alpha & \delta & & & \\ \gamma & \alpha & \delta & & \\ & \ddots & \ddots & \ddots & \\ & & \gamma & \alpha & \delta \\ & & & \gamma & \alpha \end{bmatrix} \tag{1}$$

and we wish to find its eigenvalues and eigenvectors. To begin with, let us just consider the special case where $\alpha = 2$, $\gamma = \delta = -1$, that is

$$\hat{A} = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix} \tag{2}$$

The approach we use, is to consider the eigenvalue equation $\hat{A}x = \lambda x$ on component form

$$-x_{k-1} + (2-\lambda)x_k - x_{k+1} = 0, \quad k = 1, \ldots, n, \tag{3}$$

if we adopt the convention that $x_0 = x_{n+1} = 0$. From the theory of linear constant coefficient difference equations we make an ansatz that solutions are of the form $x_k = r^k$ for some $r \in \mathbb{C}$. Substituting this ansatz into (3) and discarding the trivial 0-solution, we obtain the following quadratic equation for r:

$$r^2 - (2-\lambda)r + 1 = 0, \tag{4}$$

whose root we denote r_1, r_2. We then observe that one must have $r_1 r_2 = 1$ since the product of the roots is the constant term divided by the leading coefficient. So the general solution to (3) can be written

$$x_k = C r_1^k + C' r_2^k$$

for constants C, C'. Now we use the boundary conditions $x_0 = x_{n+1} = 0$ to get

$$r_1^{n+1} - r_2^{n+1} = 0$$

which, using $r_1 r_2 = 1$ implies $r_1^{2n+2} = 1$, so r_1 is found by taking the $(2n+2)th$ roots of unity and the corresponding $r_2 = \overline{r}_1$

$$r_{1,m} = \mathrm{e}^{\mathrm{i}\frac{m\pi}{n+1}}, \; m = 1, \ldots, n.$$

Notice that taking $m = n+1$ would result in the 0-solution once again, and the case $m > n+1$ can be discarded due to the symmetry in the solutions for r_1 and r_2. Being primarily interested in the corresponding eigenvalues λ_m we observe from (4) that $r_{1,m} + r_{2,m} = 2\,\mathrm{Re}\,(r_{1,m}) = 2 - \lambda_m = 2\cos\frac{m\pi}{n+1}$. So

$$\lambda_m = 2 - 2\cos\frac{m\pi}{n+1} = 4\sin^2\frac{m\pi}{2(n+1)}, \quad m = 1, \ldots, n. \tag{5}$$

Suppose now that we want to compute the eigenvalues of the more general matrix (1). The following identity, valid for any matrix $M \in \mathbb{C}^{n\times n}$ is obvious

$$Mx = \lambda x \quad \Leftrightarrow \quad (aM + cI)x = (a\lambda + c)x, \quad a, c \in \mathbb{C}. \tag{6}$$

so that if λ is an eigenvalue of M, then $a\lambda+c$ is an eigenvalue of $aM+cI$ The second tool we use is a similarity transform; let $X = \mathrm{diag}(1, \mu, \mu^2, \ldots, \mu^{n-1}),\ 0 \neq \mu \in \mathbb{C}$. Then for A as in (1) compute

$$B = X^{-1}AX = \begin{bmatrix} \alpha & \delta\mu & & & \\ \gamma/\mu & \alpha & \delta\mu & & \\ & \ddots & \ddots & \ddots & \\ & & \gamma/\mu & \alpha & \delta\mu \\ & & & \gamma/\mu & \alpha \end{bmatrix} \tag{7}$$

Suppose for simplicity that $\delta\gamma > 0$. Choosing $\mu = \sqrt{\frac{\gamma}{\delta}}$, the effect is that B then is a tridiagonal symmetric matrix with off-diagonal element $\sqrt{\gamma\delta}$ and diagonal elements α. Also recall that B has the same eigenvalues as A. We now just need to observe that

$$B = -\sqrt{\gamma\delta}\hat{A} + (\alpha + 2\sqrt{\gamma\delta})\, I$$

such that

$$\lambda_m(A) = -\sqrt{\gamma\delta}\lambda_m(\hat{A}) + \alpha + 2\sqrt{\gamma\delta}$$

Using (5) and (6) we get

$$\lambda_m(A) = \lambda_m(B) = \alpha + 2\sqrt{\gamma\delta}\cos\frac{m\pi}{n+1},\ m = 1, \ldots, n.$$

It is left as an exercise to the reader to find the corresponding formula when $\gamma\delta < 0$.