TMA4205 Numerical Linear Algebra Fall 2013

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Exercise set 2

1 We shall study some properties of positive definite matrices.

a) Prove that the following two statements are equivalent for a normal matrix $A \in \mathbb{R}^{n\times n}$:

- The matrix A is positive definite.
- All the eigenvalues of A are positive.

b) Prove that the condition number based on the Euclidean norm of a normal matrix A, can be written as

$$\kappa(A) = \frac{\max_i |\lambda_i|}{\min_i |\lambda_i|},$$

where λ_i are the eigenvalues of A.

c) Prove that a matrix A is positive definite if and only if A^{-1} is positive definite.

d) Let $A \in \mathbb{R}^{n\times n}$ and let $x \in \mathbb{R}^n$. The expression

$$R(x) = \frac{x^{\mathrm{T}} A x}{x^{\mathrm{T}} x}$$

is called the Rayleigh quotient of A. If A is normal and positive definite, show that

$$\lambda_1 \le R(x) \le \lambda_n \quad \text{for all } x \in \mathbb{R}^n \setminus \{0\},$$

where λ_1 and λ_n are the smallest and largest eigenvalue of A respectively.

2 Consider the matrix

$$A = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}.$$

a) Is A normal?

b) Find the eigenvalues and eigenvectors of A.

c) Are the eigenvectors linearly independent? Are they orthogonal?

d) Can A be diagonalized?

e) Show that there exists a constant $\alpha > 0$ so that $u^{\mathrm{T}} A u \ge \alpha \|u\|_2^2$ for all $u \in \mathbb{R}^2$. What is the largest possible value for α?

f) Is A positive definite?

g) Find a Schur factorization for A.

$\boxed{3}$ We consider again the Poisson problem

$$
\begin{aligned}
-\Delta u &= f, &&\text{in } \Omega = (0,1)\times(0,1),\\
u &= 0, &&\text{on } \partial\Omega.
\end{aligned}
$$

We discretize the system on a uniform grid with step-size $h = 1/n$ in each direction.

a) Solve the resulting system of linear equations in MATLAB with three different methods:

- **i)** The diagonalization method discussed in Einar Rønquist's note.
- **ii)** LU-factorization with sparse matrices.
- **iii)** Full LU-factorization without sparse matrices.

b) Do the timings scale as expected?

c) Calculate the Euclidean condition number $\kappa(A)$ of the discrete Laplacian A for various different values of n. How does $\kappa(A)$ scale with n?

d) How does $\kappa(A)$ scale for the one-dimensional Laplacian? Compare these results with the results from question **c)**.