Norwegian University of Science and Technology
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TMA4205 Numerical Linear Algebra Fall 2013

Exercise set 4

1 **Saad, Exercise 5.3** In Section 5.3.3, it was shown that using a one-dimensional projection method with $\mathcal{K} = \text{span}\{A^T r\}$ and $\mathcal{L} = \text{span}\{AA^T r\}$ is equivalent to using the steepest descent method on the normal equations $A^T Ax = A^T b$.

Show that an *orthogonal* projection method for $A^T Ax = A^T b$ with search space $\mathcal{K}$ is equivalent to applying a projection method onto $\mathcal{K}$ orthogonally to $\mathcal{L} = A\mathcal{K}$ for the problem $Ax = b$.

2 Algorithm 6.1 in Saad is implemented in the MATLAB-function `arnoldi_gs.m`. This algorithm constructs an orthogonal basis for the Krylov subspace $\mathcal{K}_m(A, v)$ based on a classical Gram–Schmidt procedure. Test this function on the matrix A generated by `poisson2.m` (use sparse matrices) for different values of m and $N = n^2$. For instance, choose $N = 100$, $v = e_1$, and $m = 10, 20, 30, 40, 50$.

a) Test to what extent the relation $V_m^T AV_m = H_m$ from Proposition 6.5 in Saad is fulfilled. Also check if the vectors $v_1, \dots, v_m$ really are orthonormal, i.e. check whether $V_m^T V_m = I_m$ (exactly).

b) Modify the function `arnoldi_gs.m` such that it uses modified Gram–Schmidt. Repeat the experiments from the previous question.

3 **a)** If A is symmetric and positive definite (SPD), show that A^{-1} can be used to define a norm on $\mathbb{R}^n$,

$$\|v\|_{A^{-1}} = (v^T A^{-1} v)^{1/2}$$

b) We know that the conjugate gradient (CG) method will minimize the error in A-norm over all elements in the Krylov subspace $\mathcal{K}_m(A, r_0)$. Show that the algorithm also, in each iteration, will minimize the associated residual in A^{-1}-norm.

c) Each update of the solution in CG can be expressed as $x_{j+1} = x_j + \alpha_j p_j$, where $\alpha_j = (r_j, r_j)/(Ap_j, p_j)$ (see Algorithm 6.18 in Saad). Show that α_j is optimal in the sense that it minimizes the functional $f(w) = \frac{1}{2} w^T Aw - w^T b$, $f : \mathbb{R}^n \to \mathbb{R}$, along the search direction p_j.