1 In this exercise we consider the restriction operator I_h^{2h} (based on full weighting) used in the multigrid algorithm. We discussed this operator in class in the context of solving the one-dimensional Poisson problem.

a) What is the rank of this operator?

b) **Briggs, Henson & McCormick, Exercise 3.5** Show that the null space of the restriction operator I_h^{2h} has a basis consisting of vectors of the form

$$(0,\dots,0,-1,2,-1,0,\dots,0)^{\mathrm{T}}.$$

Count the number of such vectors and show that $\dim\big(\operatorname{Ker} I_h^{2h}\big) = n/2$.

c) Do the basis vectors for the null space of I_h^{2h} correspond to oscillatory modes?

d) Use MATLAB to compute the SVD of I_h^{2h} in the case $n = 1/h = 8$, i.e., $I_h^{2h} \in \mathbb{R}^{3\times 7}$; use the built-in routine for this. Is the result from **a)** consistent with the result from the SVD? What does the SVD say about $\dim\big(\operatorname{Ker} I_h^{2h}\big)$?

2 Consider the two-dimensional Poisson problem

$$\begin{cases} -\nabla^2 U = f & \text{in } \Omega = (-1,1)\times(0,1), \\ U = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f(x,y) = \pi^2\big(1-5\cos(2\pi y)\big)\sin(\pi x)$. We discretize this problem using the 5-point finite difference method on a uniform grid denoted by $x_i = -1+ih$, $y_j = jh$ with $i = 0,\dots,2N$, $j = 0,\dots,N$ and $h = 1/N$. The discrete system of equations can be expressed as $Au = b$, where A is the discrete Laplace operator, u is the unknown vector, and b is the known right-hand side. In the implementation, the matrix A should not be explicitly constructed. Instead, make a function that calculates the result of multiplying A with a given vector, without forming A.

a) Solve the system $Au = b$ using the preconditioned conjugate gradient method. You should implement your own version of this algorithm. Set the preconditioner to be the identity operator, i.e., consider first the unpreconditioned case. Plot the 2-norm of the residual vector as a function of the iteration number in a semi-log plot.

b) Use the diagonal of A as a preconditioner. Does this help the convergence rate? Explain your findings.

The domain is now decomposed into two equal halves, Ω_1 an Ω_2 as shown in Figure 1, and the overlapping domains $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ overlap by an amount $\delta = h$.

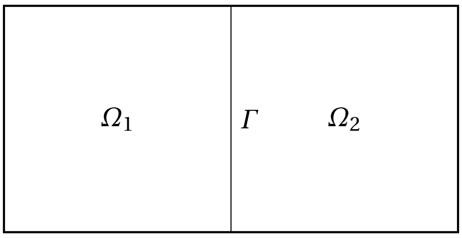

Figure 1: The domain Ω decomposed into two equal subdomains Ω_1 and Ω_2. The overlapping subdomains $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ overlap at Γ with an overlap amount $\delta = h$.

c) Use the additive overlapping Schwarz preconditioner, plot the convergence behavior and discuss your findings.

d) Use the multiplicative overlapping Schwarz preconditioner, plot the convergence behavior and discuss your findings.

e) Compare the number of iterations of the four different preconditioners and comment.