Norwegian University of Science and Technology
Department of Mathematical Sciences

Contact during exam:
Elena Celledoni, tlf. 93541

EXAM IN TMA4205 NUMERICAL LINEAR ALGEBRA

Friday December 8, 2006
Time: 09:00–13:00

Aids: A – Alle printed and hand written aids are allowed.
All calculators are allowed.

Problem 1

a) Show that the inverse of the matrix

$$I - \mathbf{u}\mathbf{v}^T$$

where I is the $n \times n$ identity matrix and $\mathbf{u}, \mathbf{v} \in \mathbf{R}^n$ and $\mathbf{v}^T\mathbf{u} \neq 1$, is of the type

$$I + \gamma \mathbf{u}\mathbf{v}^T.$$

Find γ.

b) Estimate the condition number $\mathcal{K}_2(I - \mathbf{u}\mathbf{v}^T)$ by using $\|\mathbf{u}\|_2$ and $\|\mathbf{v}\|_2$. Suppose $\mathbf{v}^T\mathbf{u} \neq 1$.

c) Suppose that we are solving the $n \times n$ linear system

$$A\mathbf{x} = \mathbf{b}, \qquad A = B - \mathbf{w}\mathbf{z}^T, \quad \mathbf{z} \in \mathbf{R}^n, \mathbf{w} = B\mathbf{z}$$

where B stems from the discretization of a Laplacian, for instance, by the finite difference or the finite element method. Suppose that n is large, that B is invertertible, and that we can use a multigrid V- or W-cycle for efficient solution of linear systems of the form $B\mathbf{y} = \mathbf{c}$. We shall therefore use B^{-1} as preconditioner in our problem.

Find the conditions that $\mathbf{z}$ must fulfill in order to guarantee the convergence of the conjugate gradient method when applied to the preconditioned system

$$B^{-1}A\mathbf{x} = B^{-1}\mathbf{b}.$$

Suppose that $\|\mathbf{z}\|_2 \leq 0.5$, and use the convergence estimate for the conjugate gradient algorithm and the estimate of $K_2(I - \mathbf{z}\mathbf{z}^T)$ for finding the minimal number of iterations necessary to guarantee that

$$\frac{\|\mathbf{x} - \mathbf{x}_m\|_{B^{-1}A}}{\|\mathbf{x} - \mathbf{x}_0\|_{B^{-1}A}} \leq 10^{-3}.$$

d) Use the result from **a**) and find an algorithm for solving $A\mathbf{x} = \mathbf{b}$ that works for every $\mathbf{z}$ and $\mathbf{w}$ such that A is invertible.

e) Suppose that B is an $n \times n$ matrix that stems from a discretization of the Helmholtz equation with periodic boundary conditions, i.e.

$$\alpha u(x) + \Delta u(x) = \psi(x), \quad -\pi \leq x \leq \pi, \quad u(-\pi) = u(\pi), \quad 0 < \alpha \leq 1,$$

where Δ is the Laplacian. After discretization with the spectral method we get

$$B = \tilde{\Omega}^H \Lambda \tilde{\Omega}, \qquad \tilde{\Omega}^H \tilde{\Omega} = I,$$

where Λ is a diagonal matrix. We suppose that n is an even integer. The diagonal of Λ is

$$[\alpha, \alpha, \alpha + 1, \alpha + 1, \ldots, \alpha + (k-1)^2, \alpha + (k-1)^2, \alpha + k^2, \alpha + k^2],$$

with $k = n/2 - 1$. The unitary matrix $\tilde{\Omega}$ is such that $\tilde{\Omega} = P\Omega P^T$ where P is a permutation matrix, and Ω is the Fourier matrix. Show that the diagonal elements in the matrix B is

$$B_{j,j} = \frac{2 \cdot \alpha}{n} + \frac{n^2 - 3n + 2}{12}, \quad j = 1, \ldots, n.$$

Hint. Note that the matrix $\tilde{B} = \Omega^H(P^T \Lambda P)\Omega$ is cyclic, symmetric. Find the diagonal elements of $\tilde{B}$. Show that $\tilde{B}$ and B have the same diagonal elements.

Given. A permutation matrix is a matrix obtained by permuting the rows or columns of the identity matrix.
The Fourier matrix Ω has elements

$$\Omega_{p,l} = \frac{1}{\sqrt{n}} \exp\left(i \cdot \frac{2\pi}{n}(p-1)(l-1)\right), \quad i = \sqrt{-1}, \quad p, l = 1, \ldots, n.$$

The eigenvalues of a cyclic matrix are the components of the vector

$$\mathbf{g} = \sqrt{n} \cdot \Omega^H \tilde{\mathbf{b}},$$

where $\tilde{\mathbf{b}}^T$ is the first row in $\tilde{B}$.
Remember that

$$\sum_{l=1}^{m} l^2 = \frac{m(m+1)(2m+1)}{6}.$$

f) Consier the weighted Jacobi-iteration for solving $B\mathbf{y} = \mathbf{c}$, d.v.s.

$$\mathbf{y}^{m+1} = (1-\omega)\mathbf{y}^m + \omega D^{-1}(E+F)\mathbf{y}^m + \omega D^{-1}\mathbf{c},$$

$B = D - E - F$ where D is diagonal and E is lower triangular, and F upper triangular, and $0 < \omega \leq 1$ is the relaxation parameter.

Show that the iteration can be written as

$$\mathbf{y}^{m+1} = G_\omega \mathbf{y}^m + \omega D^{-1}\mathbf{c},$$

and use this to show that the eigen values of G_ω are

$$\mu_j = 1 - \omega \frac{12 \cdot \lambda_j}{n^2 - 3n + 2 + 24\alpha/n}, \quad j = 1, \dots, n$$

where λ_j are the eigenvalues of B, i.e.

$$\lambda_j = \begin{cases} \alpha + (\frac{j}{2} - 1)^2, & \text{if } j \text{ is even,} \\ \alpha + (\frac{j+1}{2} - 1)^2, & \text{if } j \text{ is odd.} \end{cases}$$

g) Investigate the smoothing properties of weighted Jacobi. Express the initial error $\mathbf{e}^0 = \mathbf{y} - \mathbf{y}^0$ as

$$\mathbf{e}^0 = \sum_{j=1}^{n} f_j w_j,$$

where w_j are the columns of the matrix $\tilde{\Omega}$. Find the corresponding formula for the error $\mathbf{e}^m = \mathbf{y} - \mathbf{y}^m$ by using the coefficients f_j and the eigenvalues and eigenvectors of G_ω. Determine ω which yields the best damping of the high frequency error modes from the condition $-\mu_{n/2} = \mu_n$.

Problem 2

We shall consider sensitivity with respect to rounding error in the system $AXC = B$ where A is a real $n \times n$ invertible matrix, X is a real $n \times p$ matrix, C is a real $p \times p$ invertible matrix and B is an $n \times p$ matrix, with $n \geq p$. Consider the perturbed system

$$(A + \varepsilon\Delta A)X(\varepsilon)(C + \varepsilon\Delta C) = B + \varepsilon\Delta B.$$

Find an upper bound for the relative error,

$$\frac{\|X(\varepsilon) - X\|_2}{\|X\|_2},$$

by using the relative error in input data, A, B og C and the condition numbers of A and C.

Problem 3

Consider the Arnoldi algorithms for computing an orthonormal basis of the Krylov subspace

$$K_m(A, \mathbf{u}_0) = \text{span}\{\mathbf{u}_0, A\mathbf{u}_0, \ldots, A^{m-1}\mathbf{u}_0\},$$

where A is an $n \times n$ matrix and $\mathbf{u}_0 \in \mathbf{R}^n$. The eigenvalues of the Arnoldi upper Hessenberg matrix, $V_m^T A V_m = H_m$, can be computed efficiently for instance by using a shifted QR iteration algorithm. Explain why. Suppose that ν_k is an eigenvalue of H_m and $\mathbf{y}_k \in \mathbf{R}^m$ is the corresponding normalized eigenvector. Consider ν_k as an approximation to an eigenvalue of A, and $V_m\mathbf{y}_k$ as an approximation to the corresponding eigenvector. Find an error bound for

$$\|AV_m\mathbf{y}_k - \nu_k V_m \mathbf{y}_k\|_2.$$

Use known properties of the Arnoldi algorithm for this purpose.