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Contact during exam
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EXAM IN NUMERICAL LINEAR ALGEBRA (TMA4205)

Friday December 5, 2008
Time: 09:00–13:00

Aids: Category A, All printed and hand written aids allowed. All calculators allowed.

Problem 1 The partial differential equation

$$-u_{xx} + c\,u = f, \quad c \geq 0.$$

with homogeneous Dirichlet boundary conditions, yields after discretizing with centered differences, a linear system of the form $Au = b$ where $A = \mathrm{tridiag}(-1, 2+\gamma, -1)$, $A \in \mathbb{R}^{m\times m}$, $\gamma = c/(m+1)^2$. We find that A has eigenvalues

$$\lambda_k = \gamma + 4\sin^2\left(\frac{k\pi}{2(m+1)}\right), \quad k = 1, \ldots, m,$$

and corresponding eigenvectors

$$w_k = \begin{bmatrix} \sin\left(\frac{k\pi}{m+1}\right) \\ \sin\left(\frac{2k\pi}{m+1}\right) \\ \vdots \\ \sin\left(\frac{mk\pi}{m+1}\right) \end{bmatrix}.$$

a) Formulate the weighted Jacobi method with relaxation parameter ω for this linear system,and show that the iteration can be written in the form

$$u^{(q+1)} = G_\omega\, u^{(q)} + \frac{\omega}{2+\gamma}\, b \quad \text{where} \quad G_\omega = I - \frac{\omega}{2+\gamma} A,$$

and that the iteration matrix G_ω has eigenvalues

$$\mu_k = 1 - \frac{\omega}{2+\gamma}\left(\gamma + 4\sin^2\left(\frac{k\pi}{2(m+1)}\right)\right), \quad k = 1, \ldots, m,$$

and the same eigenvectors as A

b) The error after q iterations with weighted Jacobi on this system can be written as

$$e^{(q)} = G_\omega^q \, e^{(0)} = \sum_{k=1}^{m} \rho_k \mu_k^q w_k,$$

where

$$e^{(0)} = \sum_{k=1}^{m} \rho_k w_k$$

is the initial error $e^{(0)}$ expressed in terms of the eigenvectors w_k.

Determine the (optimal) value ω^{opt} of ω from which the best damping occurs of the upper half of the spectrum of the error, i.e. find

$$\omega^{\text{opt}} = \arg\min_{\omega} \max_{k > \frac{m+1}{2}} |\mu_k|.$$

Verify that you get back to the known $\omega^{\text{opt}} = 2/3$ when $\gamma = 0$. What happens when γ tends to infinity?

Problem 2

a) Describe briefly the idea behind the projection methods for solving linear systems $Ax = b$. Use approximately 4-5 lines.

Assume in the rest of this problem that $A \in \mathbb{R}^{n\times n}$ is symmetric positive definite, $b \in \mathbb{R}^n$ is a given right hand side, and $x = A^{-1}b$. We let $r_k = b - Ax_k$ og $e_k = x - x_k = A^{-1}r_k$, for $k \geq 0$ and assume that x_0 is a given vector.

b) Let us use as approximation space $\mathcal{K} = \text{span}\{v\}$ and constraint space $\mathcal{L} = \mathcal{K}$. Let x_1 be the result of one step with the projection method. Show that

$$\langle Ae_1, e_1\rangle = \langle Ae_0, e_0\rangle - \langle r_0, v\rangle^2 / \langle Av, v\rangle.$$

c) The method from the previous question says nothing about how the new search direction v is chosen in each iteration. Let us therefore introduce the following principle: Choose v_k such that $\langle v_k, r_k\rangle = \|r_k\|_1$, that is, let the components in v_k be 1 and -1, negative (-1) if the corresponding component in r_k is negative, and positive $(+1)$ if the r_k component is ≥ 0. Show that

$$\|e_{k+1}\|_A \leq \left(1 - \frac{1}{n\kappa(A)}\right)^{1/2} \|e_k\|_A.$$

Here $\kappa(A) = \|A\|_2\|A^{-1}\|_2$ while $\|w\|_A = \langle Aw, w\rangle^{1/2}$.

Problem 3 Let the matrix A be given as

$$A = \begin{bmatrix} 8 & 1 & 0 \\ 1 & 4 & \varepsilon \\ 0 & \varepsilon & 1 \end{bmatrix}, \qquad |\varepsilon| \leq 1.$$

a) Give an estimate for the eigenvalues of A by using the Gerschgorin theorem. In particular, what can one say about the smallest eigenvalue? Make a sketch to illustrate.

b) Show for instance by using a suitable diagonal similarity transformation the sharper estimate $|\lambda_3 - 1| \leq \varepsilon^2$ for the smallest eigenvalue of A.

c) For $\varepsilon = 0.1$ one has found Q and R such that $A - I = QR$ where

$$Q = \begin{bmatrix} -0.9899 & 0.1413 & 0.0050 \\ -0.1414 & -0.9893 & -0.0350 \\ 0 & -0.0353 & 0.9994 \end{bmatrix}, \quad R = \begin{bmatrix} -7.0711 & -1.4142 & -0.0141 \\ 0 & -2.8302 & -0.0989 \\ 0 & 0 & -0.0035 \end{bmatrix}.$$

Find an approximation to the smallest eigenvalue of A from this.

Problem 4

a) Find the singular value decomposition of the matrix

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}.$$

b) The matrix in the previous question is a special case of a matrix $B \in \mathbb{R}^{(n+1)\times n}$ where $B_{k,k} = 1,\ B_{k+1,k} = -1$ for $k = 1, \ldots, n$ and where all other elements of B are zero. Determine the singular value decomposition of B.