Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4205 Numerical Linear Algebra Fall 2013

Semester project – part 1

$\boxed{1}$ This assignment is dedicated entirely to linear algebra aspects of the solution of the tridiagonal, linear system $Au = b$ arising from the convection-diffusion equation

$$\begin{aligned} -U_{xx} + aU_x = f, \quad &\text{in } \Omega = (0,1) \\ U(0) = 1, \quad &U(1) = -1, \end{aligned} \tag{1}$$

where $a = a(x)$ and $f = f(x)$ are given functions. We discretize the problem on a uniform grid with step-size $h = 1/n$, where the grid points are $x_j = jh$, $j = 0, \dots, n$. We shall achieve this by using the second-order centered-difference schemes

$$U_{xx}(x_j) \approx \frac{u_{j-1} - 2u_j + u_{j+1}}{h^2}, \qquad U_x \approx \frac{u_{j+1} - u_{j-1}}{2h},$$

where $u_j \approx U(x_j)$ represents the numerical approximation of the solution at the grid point x_j. The resulting linear system can be written as $Au = b$, where A and b are of the form

$$A = \begin{bmatrix} \alpha & \delta_1 & & & \\ \gamma_2 & \alpha & \delta_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \gamma_{n-2} & \alpha & \delta_{n-2} \\ & & & \gamma_{n-1} & \alpha \end{bmatrix}, \qquad b = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{n-2} \\ \beta_{n-1} \end{bmatrix},$$

and $u \in \mathbb{R}^{n-1}$ is the vector of unknowns. Let β_j be expressed as $\beta_j = h^2 f(x_j) + \tau_j$, where τ_j accounts for the boundary contributions.

a) Suppose $a(x) = 2x$. Write down the expressions for α, δ_j, γ_j and τ_j.

From now on we assume $a = 2$, so $\delta = \delta_j$ and $\gamma = \gamma_j$ are constant for each j.

b) Give an explicit formula for the eigenvalues of A.

Hint: Use the note “Eigenvalues of tridiagonal Toeplitz matrices”, which can be found on the home page. No derivations are required.

c) We split the matrix into two pieces $A = M - N$, where M is the diagonal part of A. What are the eigenvalues of the matrix N?

d) We wish to solve the discrete system with a simple Jacobi iteration. Show that this can be expressed as $u^{(k+1)} = Gu^{(k)} + M^{-1}b$, where $G = M^{-1}N$ is the iteration matrix. What are the eigenvalues of G? What is the spectral radius of G? What does Gershgorin’s theorem say about the eigenvalues of G? Can this theorem be used to predict the convergence of the Jacobi iteration? Explain.

e) How would you expect the error $e^{(k)} = u - u^{(k)}$ to behave as a function of k and n? In other words, if you double n, what must you do with k in order to get close to the same error $e^{(k)}$?

Hint: First show that the matrix G is *almost* normal. The higher n is, the closer to normal it gets.

f) Consider again the problem (1) with exact solution given by $U(x) = \cos(\pi x)$. What is the corresponding right-hand side f? Let $n = 20$ and use Jacobi iteration to solve the corresponding discrete system with this choice of f. Define u_* to be the vector with entries $U(x_i)$, $i = 1, \ldots, n-1$, i.e. the continuous solution evaluated at the interior grid points. Define also $e_*^{(k)} = u_* - u^{(k)}$ and plot $\log(\|e_*^{(k)}\|_\infty)$ as a function of k. Iterate until the error $e_*^{(k)}$ no longer changes. Next, increase n to 40, and repeat the solution process. Finally, do it with $n = 80$. Compare the convergence behaviour for all three cases (e.g. in one single plot). Are the results as expected? Can you explain your observations?

Hint: $u_* - u^{(k)} = (u_* - u) + (u - u^{(k)})$

g) What do you think the convergence behaviour would have been if we had plotted instead $\log(\|e^{(k)}\|_\infty)$ as a function of k?

h) Can you suggest a stopping criterion for the Jacobi iteration?