Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4205 Numerical Linear Algebra Fall 2013

Solutions to exercise set 2

1 **a)** Since A is normal, it can be orthogonally diagonalized such that $A = Q\Lambda Q^{\mathrm{T}}$, where the columns of Q are the eigenvectors q_i, and the entries of Λ are the eigenvalues λ_i of A.

First, assume that $x^{\mathrm{T}}Ax > 0$ for all $x \neq 0$. This must be true for $x = q_i$, so $q_i^{\mathrm{T}}Aq_i = \lambda_i q_i^{\mathrm{T}} q_i = \lambda_i \|q_i\|_2^2 > 0$. We have proved that if A is positive definite, then all the eigenvalues are positive.

Next, assume that the eigenvalues of A are positive. Since the eigenvectors q_i form a basis for $\mathbb{R}^n$, we can represent any vector as a linear combination of the q_i. Let $u = Qc$ be any nonzero vector in $\mathbb{R}^n$. Then $u^{\mathrm{T}}Au = c^{\mathrm{T}}Q^{\mathrm{T}}Q\Lambda Q^{\mathrm{T}}Qc = c^{\mathrm{T}}\Lambda c = \sum_i \lambda_i c_i^2$. All the λ_i are positive, and at least one of the c_i are nonzero, so $u^{\mathrm{T}}Au > 0$. Thus, we have proved the other direction.

b) The matrix A is normal, so it can be unitarily diagonalized, $A = Q\Lambda Q^{\mathrm{H}}$. Thus, $AA^{\mathrm{H}} = Q\Lambda\Lambda^{\mathrm{H}}Q^{\mathrm{H}} = Q\Lambda^{\mathrm{H}}\Lambda Q^{\mathrm{H}} = A^{\mathrm{H}}A$. We can then calculate the condition number:

$$\kappa(A) = \|A\|_2\|A^{-1}\|_2 = \sqrt{\rho(AA^{\mathrm{H}})}\sqrt{\rho(A^{-1}A^{-\mathrm{H}})} = \max_i|\lambda_i|\max_i\frac{1}{|\lambda_i|} = \frac{\max_i|\lambda_i|}{\min_i|\lambda_i|}.$$

c) Assume first that A is positive definite. Then A is non-singular, so given a vector $y \neq 0$, we can find a unique $x \neq 0$ so that $y = Ax$. Thus, $y^{\mathrm{T}}A^{-1}y = x^{\mathrm{T}}A^{\mathrm{T}}x = x^{\mathrm{T}}Ax > 0$. To prove the converse, just substitute $A \leftrightarrow A^{-1}$.

d) Since A is normal, we have that $A = Q\Lambda Q^{\mathrm{T}}$ with $Q^{\mathrm{T}}Q = QQ^{\mathrm{T}} = \mathrm{I}$. Hence, we can express the Rayleigh quotient as

$$R(x) = \frac{x^{\mathrm{T}}Ax}{x^{\mathrm{T}}x} = \frac{x^{\mathrm{T}}Q\Lambda Q^{\mathrm{T}}x}{x^{\mathrm{T}}QQ^{\mathrm{T}}x}.$$

If we now define $y = Q^{\mathrm{T}}x$, we have

$$R = \frac{y^{\mathrm{T}}\Lambda y}{y^{\mathrm{T}}y} = \frac{\sum_{i=1}^n \lambda_i y_i^2}{\sum_{i=1}^n y_i^2},$$

and using the fact that all the eigenvalues of A are positive, we obtain the bounds

$$R = \frac{\sum_{i=1}^n \lambda_i y_i^2}{\sum_{i=1}^n y_i^2} \leq \frac{\lambda_n \sum_{i=1}^n y_i^2}{\sum_{i=1}^n y_i^2} = \lambda_n, \qquad R = \frac{\sum_{i=1}^n \lambda_i y_i^2}{\sum_{i=1}^n y_i^2} \geq \frac{\lambda_1 \sum_{i=1}^n y_i^2}{\sum_{i=1}^n y_i^2} = \lambda_1.$$

Here, λ_1 is the smallest and λ_n is the largest eigenvalue of A.

2 **a)** A is normal if $AA^{\mathrm{H}} = A^{\mathrm{H}}A$. Here,

$$A = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}, \qquad A^{\mathrm{H}} = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix},$$

which leads to

$$AA^{\mathrm{H}} = \begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix} \qquad \text{and} \qquad A^{\mathrm{H}}A = \begin{bmatrix} 5 & -1 \\ -1 & 0 \end{bmatrix}.$$

Hence, A is not normal.

b) Here, A is a lower triangular matrix, and we have the eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 1$ on the diagonal. For the eigenvectors, we have that

$$Av_1 = 2v_1 \quad \Longrightarrow \quad v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad \text{and} \qquad Av_2 = v_2 \quad \Longrightarrow \quad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

c) We see that the eigenvectors are linearly independent, but they are not orthogonal since $v_1^{\mathrm{H}} v_2 \neq 0$.

d) Yes. All the eigenvalues are nondefective, so the eigenvectors form a full set of linearly independent vectors for $\mathbb{R}^2$. This implies that A can be diagonalized, i.e. we can write $A = V\Lambda V^{-1}$, where the columns of V are the eigenvectors of A and Λ is a diagonal matrix containing the eigenvalues of A.

e) We rewrite A as

$$A = \underbrace{\tfrac{1}{2}(A + A^{\mathrm{T}})}_{H} + \underbrace{\tfrac{1}{2}(A - A^{\mathrm{T}})}_{S},$$

where

$$\begin{aligned} H^{\mathrm{T}} &= H && \text{i.e. symmetric,} \\ S^{\mathrm{T}} &= -S && \text{i.e. skew-symmetric.} \end{aligned}$$

Since H is symmetric, we can orthogonally diagonalize it, $H = Q\Lambda Q^{\mathrm{T}}$. Hence, for all $u \in \mathbb{R}^2$, letting $c = Q^{\mathrm{T}}u$,

$$u^{\mathrm{T}}Au = u^{\mathrm{T}}Hu + \underbrace{u^{\mathrm{T}}Su}_{0} = u^{\mathrm{T}}Hu = c^{\mathrm{T}}\Lambda c = \sum_i \lambda_i c_i^2.$$

We find that the matrix H is

$$H = \begin{bmatrix} 2 & -1/2 \\ -1/2 & 1 \end{bmatrix},$$

which has eigenvalues $\lambda = (3 \pm \sqrt{2})/2$, which are both positive. Thus, we choose $\alpha = \lambda_{\min}(H) = (3 - \sqrt{2})/2$, which makes

$$u^{\mathrm{T}}Au = \sum_i \lambda_i c_i^2 \geq \alpha \|c\|_2^2 = \alpha \|u\|_2^2.$$

f) Yes, this follows from **e)** since $\alpha > 0$.

g) A Schur factorization is a product $A = QRQ^{\mathrm{H}}$ for Q unitary and R upper triangular. From **d)**, we know that A can be diagonalized, $A = V\Lambda V^{-1}$, however, V is not unitary, so this is not a Schur factorization. We orthonormalize V using, say, the Gram–Schmidt process, to obtain $V = Q\tilde{R}$, where Q is unitary and $\tilde{R}$ is upper triangular. Substituting this factorization into the diagonalization leads to a Schur decomposition

$$A = QRQ^{\mathrm{H}}, \qquad \text{where} \qquad R = \tilde{R}\Lambda\tilde{R}^{-1}.$$

n	Time (s)	Time$/n^3$
500	$1.30 \cdot 10^{-2}$	$1.04 \cdot 10^{-10}$
600	$2.29 \cdot 10^{-2}$	$1.06 \cdot 10^{-10}$
700	$3.52 \cdot 10^{-2}$	$1.03 \cdot 10^{-10}$
800	$5.27 \cdot 10^{-2}$	$1.03 \cdot 10^{-10}$
900	$7.27 \cdot 10^{-2}$	$1.00 \cdot 10^{-10}$

Table 1: Two-dimensional Poisson problem with diagonalization method

The Gram–Schmidt process yields

$$w_1 = v_1, \qquad q_1 = \frac{w_1}{\|w_1\|_2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$w_2 = v_2 - (v_2^{\mathrm{H}} q_1) q_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad q_2 = \frac{w_2}{\|w_2\|_2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Thus

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix},$$

giving

$$\begin{aligned} R &= Q^{\mathrm{H}} A Q \\ &= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

$\boxed{3}$ We consider again the Poisson problem

$$\begin{aligned} -\Delta u &= f, \quad \text{in } \Omega = (0,1) \times (0,1), \\ u &= 0, \quad \text{on } \partial\Omega. \end{aligned}$$

and discretize this with the five point formula using a uniform grid spacing, $h = 1/n$, in each spatial direction. The number of unknowns is therefore $(n-1)^2 \approx n^2$.

a) **i)** First we solve the Poisson problem using the diagonalization method. See Table 1.

ii) Next, we solve the same problem using LU factorization exploiting sparsity. See Table 2

iii) Finally, we solve the same problem using full LU factorization without exploiting sparsity. See Table 3.

b) All the methods perform as expected. Note the difference in absolute solution times. The diagonalization method is clearly much better than either of the other methods.

c) For the one-dimensional Laplace operator the continuous eigenfunctions and eigenvalues are (see the note by Rønquist)

$$\begin{aligned} u_j^*(x) &= \sin(j\pi x), \\ \lambda_j^* &= j^2\pi^2, \quad j = 1, \dots, \infty. \end{aligned}$$

n	Time (s)	Time/n^4
50	$1.55 \cdot 10^{-2}$	$2.47 \cdot 10^{-9}$
60	$2.97 \cdot 10^{-2}$	$2.29 \cdot 10^{-9}$
70	$5.12 \cdot 10^{-2}$	$2.13 \cdot 10^{-9}$
80	$8.34 \cdot 10^{-2}$	$2.04 \cdot 10^{-9}$
90	$1.27 \cdot 10^{-1}$	$1.93 \cdot 10^{-9}$

Table 2: Two-dimensional Poisson problem with LU factorization exploting sparsity

n	Time (s)	Time/n^6
50	0.153	$9.78 \cdot 10^{-12}$
60	0.404	$8.67 \cdot 10^{-12}$
70	0.945	$8.03 \cdot 10^{-12}$
80	2.031	$7.75 \cdot 10^{-12}$
90	4.026	$7.58 \cdot 10^{-12}$

Table 3: Two-dimensional Poisson problem with full LU factorization

From the note, we can also see that if we discretize the Laplace operator using a central difference scheme with uniform grid spacing, $h = 1/n$, we get the matrix $A^{1\mathrm{D}}$ with eigenvalues

$$\lambda_j = \frac{2}{h^2}\left(1 - \cos\left(\frac{j\pi}{n}\right)\right), \quad j = 1, \dots, n-1.$$

Hence, we expect

$$\begin{aligned}
\lambda_{\min} &= \frac{2}{h^2}\left(1 - \cos\left(\frac{\pi}{n}\right)\right) \approx \frac{1}{h^2}\frac{\pi^2}{n^2}, \\
\lambda_{\max} &= \frac{2}{h^2}\left(1 - \cos\left(\frac{(n-1)\pi}{n}\right)\right) \approx \frac{4}{h^2}, \\
&\Downarrow \\
\kappa(A^{1\mathrm{D}}) &= \frac{\lambda_{\max}}{\lambda_{\min}} \approx \frac{4}{\pi^2} n^2.
\end{aligned}$$

The condition number should scale as $\mathrm{O}(n^2)$ in 1D.

d) Consider now the two-dimensional Poisson problem

$$\begin{aligned}
-\Delta u &= f, \quad \text{in } \Omega = (0,1) \times (0,1), \\
u &= 0, \qquad\quad \text{on } \partial\Omega.
\end{aligned}$$

In the continuous case, the eigenfunctions and eigenvalues of the Laplace operator are given as

$$\begin{aligned}
u^*_{j,k}(x,y) &= \sin(j\pi x)\sin(k\pi y), \\
\lambda^*_{j,k} &= j^2\pi^2 + k^2\pi^2 = (j^2 + k^2)\pi^2, \\
j &= 1, \dots, \infty, \quad k = 1, \dots, \infty.
\end{aligned}$$

These results are obtained simply by using separation of variables.

n	$\lambda_{\min}(\hat{A}^{1D})$	$\lambda_{\min}(\hat{A}^{1D})$	$\kappa(A^{1D})$
10	$8.10 \cdot 10^{-2}$	3.92	48.37
20	$2.23 \cdot 10^{-2}$	3.98	178.06
30	$1.03 \cdot 10^{-2}$	3.99	388.81
40	$5.87 \cdot 10^{-3}$	3.99	680.62

(a) One dimension

n	$\lambda_{\min}(\hat{A}^{2D})$	$\lambda_{\min}(\hat{A}^{2D})$	$\kappa(A^{2D})$
10	$1.62 \cdot 10^{-1}$	7.84	48.37
20	$4.47 \cdot 10^{-2}$	7.96	178.06
30	$2.05 \cdot 10^{-2}$	7.98	388.81
40	$1.17 \cdot 10^{-2}$	7.99	680.62

(b) Two dimensions

Table 4: Condition numbers

Again, we discretize the Poisson problem using the five-point formula, and get the discrete Laplace operator (or matrix), A^{2D}. The eigenvalues of A^{2D} are

$$\lambda_{j,k} = \frac{2}{h^2}\left(1-\cos\left(\frac{j\pi}{n}\right)\right) + \frac{2}{h^2}\left(1-\cos\left(\frac{k\pi}{n}\right)\right),$$

so we expect that

$$\begin{aligned}
\lambda_{\min} &= \frac{2}{h^2}\left(1-\cos\left(\frac{\pi}{n}\right)\right) + \frac{2}{h^2}\left(1-\cos\left(\frac{\pi}{n}\right)\right) \approx \frac{2}{h^2}\frac{\pi^2}{n^2},\\
\lambda_{\min} = \lambda_{j,k} &= \frac{2}{h^2}\left(1-\cos\left(\frac{(n-1)\pi}{n}\right)\right) + \frac{2}{h^2}\left(1-\cos\left(\frac{(n-1)\pi}{n}\right)\right) \approx \frac{8}{h^2},\\
&\Downarrow\\
\kappa(A^{2D}) &= \frac{\lambda_{\max}}{\lambda_{\min}} \approx \frac{4}{\pi^2}n^2.
\end{aligned}$$

We compute the maximum and minimum eigenvalues for the matrices $\hat{A}^{1D} = h^2 A^{1D}$ and $\hat{A}^{2D} = h^2 A^{2D}$ by using the command `eig` in MATLAB. The condition number is found by the use of `cond`. See Table 4.

We observe that the condition number κ is the same for one and two dimensions. We also get the expected behavior for the minimum and maximum eigenvalues.