Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4205 Numerical Linear Algebra Fall 2013

Solutions to exercise set 3

1 **a)** We are given the matrix

$$A = \begin{bmatrix} 1 & -6 & 0 \\ 6 & 2 & 3 \\ 0 & 3 & 2 \end{bmatrix}.$$

We may here use Gershgorin's theorem to estimate the eigenvalues of A. This theorem states that all the eigenvalues of an $n \times n$ matrix A are located in one of the closed discs of the complex plane centered in $a_{i,i}$ having radius

$$r_i = \sum_{\substack{j=1 \\ j \neq i}}^{j=n} |a_{i,j}|, \quad i = 1, \ldots, n.$$

See Saad, Theorem 4.6. For our matrix, this is illustrated in Figure 1. The shaded square encapsulates all the circles, and thus also the eigenvalues. For our given matrix, the spectrum is $\sigma(A) = \{2.328, 1.336 \pm 5.196\mathrm{i}\}$. We see that this is in agreement with the estimate.

b) An MR iteration will converge if A is positive-definite. This is equivalent to $A + A^{\mathrm{T}}$ being positive-definite. We have that

$$A + A^{\mathrm{T}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 6 \\ 0 & 6 & 4 \end{bmatrix}$$

and this matrix has spectrum $\sigma(A) = \{-2, 2, 10\}$. Since one of the eigenvalues is negative, A cannot be positive-definite. Thus, the MR iteration is not guaranteed to converge.

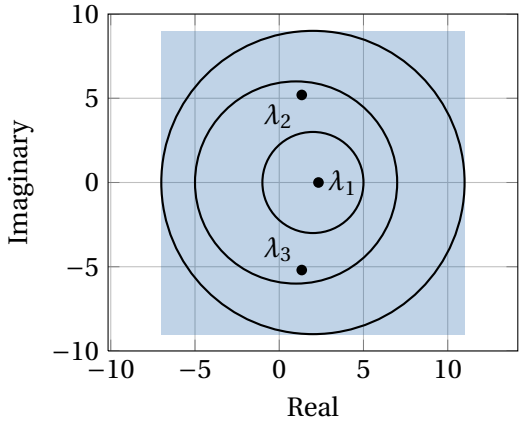

Figure 1: Exact and estimated eigenvalues of A.

2 We revisit the one-dimensional Poisson problem of exercise set 1, and we consider solving the discretized problem using Jacobi -, steepest descent (SD) -, and minimum residual (MR) iteration.

a) In general, if we have an error/residual behaviour given by

$$\|e_k\| = \rho^k \|e_0\|,$$

so

$$\log \frac{\|e_k\|}{\|e_0\|} = k \log \rho.$$

For this exercise we want to obtain an error/residual reduction by 10^{-5} such that $\log(\|e_k\|/\|e_0\|) = -5$, i.e.,

$$-5 = k \log \rho. \tag{1}$$

We will need the following useful approximations based on Maclaurin expansions:

$$\begin{aligned} \cos(x) &\approx 1 - \tfrac{1}{2}x^2, \\ \log(1+x) &\approx x, \\ (1+x)^{1/2} &\approx 1 + \tfrac{1}{2}x. \end{aligned}$$

Jacobi. For this case we have that ρ is the spectral radius of the iteration matrix, so $\rho = \cos(\pi/n) \approx 1 - \pi^2/2n^2$. From (1) and the approximation of the logarithm, we find that $k \approx 10n^2/\pi^2$.

Steepest descent. From the lectures, we know that

$$\|e_k\|_A \leq \rho^k \|e_0\|_A,$$

where

$$\rho = \frac{\lambda_{\max} - \lambda_{\min}}{\lambda_{\max} + \lambda_{\min}} = \frac{\kappa - 1}{\kappa + 1} \approx 1 - \frac{2}{\kappa}$$

for large values of $\kappa = \lambda_{\max}/\lambda_{\min}$, the condition number based on the Euclidean norm of A. From (1) and the approximation of the logarithm, we find that $k \approx 5\kappa/2$. We also have that

$$\kappa = \frac{\lambda_{\max}}{\lambda_{\min}} = \frac{\frac{2}{h^2}\left(1 - \cos\left(\frac{(n-1)\pi}{n}\right)\right)}{\frac{2}{h^2}\left(1 - \cos\left(\frac{\pi}{n}\right)\right)} \approx \frac{2}{\pi^2/2n^2} = \frac{4n^2}{\pi^2} \quad \Longrightarrow \quad k \approx \frac{10n^2}{\pi^2}. \tag{2}$$

b) We now consider MR iteration for the same problem, and wish to reduce the initial residual with 5 orders of magnitude. In this case,

$$\|r_k\|_2 \leq \rho^k \|r_0\|_2$$

with

$$\rho = \left(1 - \frac{\mu^2}{\sigma^2}\right)^{1/2},$$

where

$$\mu = \lambda_{\min}\left(\frac{A + A^{\mathrm{T}}}{2}\right), \qquad \sigma = \|A\|_2.$$

Method	Ax	αx and $x+y$	$x^{\mathrm{T}}x$
J	1	1	?
SD	1	4	2
MR	1	4	2

(a) Computational cost

Method	A	x
J	1	2
SD	1	3
MR	1	3

(b) Memory requirement

Table 1: One iteration of Jacobi, SD and MR

In our case, A is symmetric and positive-definite (SPD), and we thus get the simplified expressions

$$\mu = \lambda_{\min}(A), \qquad \sigma = \lambda_{\max}(A).$$

Hence,

$$\rho = \left(1 - \frac{\lambda_{\min}^2}{\lambda_{\max}^2}\right)^{1/2} = \left(1 - \frac{1}{\kappa^2}\right)^{1/2} \approx 1 - \frac{1}{2\kappa^2}.$$

From (1) and the approximation of the logarithm, we find that $k \approx 10\kappa^2$. We insert the expression for the condition number from (2), and find

$$k \approx \frac{160n^4}{\pi^4}.$$

We may now compare the value for k for Jacobi, SD and MR, and observe that

$$\frac{k_{\mathrm{SD/J}}}{k_{\mathrm{MR}}} \approx \frac{1}{4\kappa} = \frac{\pi^2}{16n^2}.$$

Here, $k_{\mathrm{SD/J}}$ is the number of iterations for Jacobi and steepest descent to reduce the error by 5 orders of magnitude, while k_{MR} is the number of iterations for MR to reduce the initial residual by the same amount. Since we expect κ to be large for large n, this difference is significant!

c) We now discuss the computational cost for the three iterative methods. In Table 1a we have listed the number of matrix-vector operations (vector addition and scalar multiplication) and the number of inner-products for one iteration of each method. How many floating point operations this is equivalent to, depends on the matrix A. For our Poisson problem, A is tridiagonal, and a matrix-vector product can be done in $\mathrm{O}(n)$ operations. (If A was a full matrix this would require $\mathrm{O}(n^2)$ operations.) The vector operations and inner-products also use $\mathrm{O}(n)$ operations, so the number of operations for one iteration for all three methods is $\mathcal{N}_{\mathrm{ops}}^1 \sim \mathrm{O}(n)$. Note that in a Jacobi method there is no error or residual estimate available, so the question mark in the last column is for potential error/residual estimation.

In Table 1b we indicate the memory requirement. All methods need to store enough information about A to be able to perform matrix-vector products. The sparsity of A should here be exploited, and we thus only need to store the non-zero entries, which in this case is $\mathrm{O}(n)$ (in fact, the non-zero entries in each row are the same). Both SD and MR need to store three vectors x, p and r. For Jacobi we need to store x as well as a working array. Thus, the memory requirement is $\mathrm{O}(n)$ for all methods.

We now consider the cost for k iterations. The memory requirement remains the same, while the number of operations is given by

$$\mathcal{N}_{\mathrm{ops}}^{\mathrm{tot}} = k\mathcal{N}_{\mathrm{ops}}^1,$$

Method	$\mathcal{N}_{\text{ops}}^{\text{tot}}$
J	n^3
SD	n^3
MR	n^5

Table 2: Number of floating point operations for solving the one-dimensional Poisson problem using Jacobi, SD and MR.

where $\mathcal{N}_{\text{ops}}^{1}$ is the number of floating point operations in each iteration. Using the results from **a)** and **b)**, we find the estimates in Table 2.

d) Of the three iterative methods considered here, MR is the most general one, since the requirement is only that A is positive-definite. However, we see that this method is much slower than the other two for the one-dimensional Poisson problem. Steepest descent is guaranteed to converge as long as A is SPD, while Jacobi iteration has an even stronger requirement, namely that the spectral radius of the iteration matrix must be less than 1 (which is not the case for all SPD matrices). For our particular Poisson problem, Jacobi is a little bit faster than the steepest descent, while Jacobi and steepest descent are both much faster than MR iteration.

In conclusion, a method for more general problems is typically slower than a specialized method, and for our Poisson problem, MR iteration is definitely not a good idea. We also note here that we have compared *error* reduction by 5 orders of magnitude for the first two methods, and *residual* reduction for the MR method. This is obviously not the same, but we assume that they are comparable and show the same behaviour.