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TMA4205 Numerical Linear Algebra Fall 2013

Solutions to exercise set 5

1 **a)** All the rows of the restriction operator I_h^{2h} are linearly independent, so the rank is equal to the number of rows. If $h = 1/n$, $I_h^{2h} \in \mathbb{R}^{(n/2-1)\times(n-1)}$, so $\operatorname{rank} I_h^{2h} = n/2-1$.

b) Let

$$N = \begin{bmatrix} 2 & & & & & & \\ -1 & -1 & & & & & \\ & 2 & & & & & \\ & -1 & -1 & & & & \\ & & 2 & & & & \\ & & -1 & \ddots & & & \\ & & & \ddots & -1 & & \\ & & & & 2 & & \\ & & & & -1 & -1 & \\ & & & & & 2 & \end{bmatrix} \in \mathbb{R}^{(n-1)\times(n/2)}$$

It is straight-forward to check that $AN = 0$ and that the columns are linearly independent. The number of column vectors is $n/2$. This agrees with the rank-nullity theorem, which says that

$$\dim\left(\operatorname{Ker} I_h^{2h}\right) = (n-1) - \operatorname{rank} I_h^{2h} = \frac{n}{2}.$$

Thus, the columns of N form a basis for $\operatorname{Ker} I_h^{2h}$.

c) Each of the basis vectors corresponds to a pulse of high oscillation which will get smoothed to zero by the restriction operator, so the answer is yes.

d) Using the `svd` command in MATLAB, we get a diagonal matrix with three non-zero singular values, i.e. the rank is 3. This is consistent with **a)**, where we found that the rank is $n/2-1 = 8/2-1 = 3$. Using the rank-nullity theorem we can calculate $\dim\left(\operatorname{Ker} I_h^{2h}\right)$ as in **b)**. We get $\dim\left(\operatorname{Ker} I_h^{2h}\right) = 4$.

2 We consider the discretization of the two-dimensional Poisson problem

$$-\nabla^2 U = f \quad \text{in } \Omega = (-1,1)\times(0,1), \tag{1}$$

$$U = 0 \quad \text{on } \partial\Omega, \tag{2}$$

using the 5-point finite difference method on a uniform grid denoted by $x_i = -1 + ih$, $y_j = jh$ with $i = 0,\ldots,2N$, $j = 0,\ldots,N$ and $h = 1/N$. The discrete system of equations $Au = b$ that results has dimension $n = (N-1)(2N-1) = O(N^2)$ for $N \gg 1$. The condition number is given by $\kappa = \kappa_2(A) = O(N^2)$ (see Solutions to exercise set 2).

Figure 1: Convergence behaviour of CG applied to the 2D Poisson problem (1)–(2) with $N = 100$.

Suppose e_0 and e_m denote the initial error and the error after m iterations of the conjugate gradient method (CG), and that we want to reduce the error by, say, 10 orders of magnitude. Then we have

$$\|e_m\|_A \leq 2\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^m \|e_0\|_A,$$

where $\|\cdot\|_A$ is the norm with respect to the A-inner-product. We want to have (after m iterations)

$$\frac{\|e_m\|_A}{\|e_0\|_A} \leq 2\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^m = 10^{-10}.$$

This implies that

$$\begin{aligned} \log(10^{-10}) &= 2\log\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^m \\ &\Downarrow \\ -5\log 10 &= m\log\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right). \end{aligned}$$

Taylor series expansion gives

$$\begin{aligned} -5\log 10 &= m\log\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right) \\ &\approx -\frac{2m}{\sqrt{\kappa}} \qquad \text{for } \kappa \gg 1, \\ &\Downarrow \\ m &\approx \tfrac{5}{2}\log(10)\sqrt{\kappa} = \mathrm{O}(\sqrt{\kappa}). \end{aligned}$$

Thus, we would require at least $m = \mathrm{O}(\sqrt{\kappa}) = \mathrm{O}(N)$ iterations.

In this exercise we test CG on the discrete Poisson problem in 2D. Computational tests are run for the unpreconditioned system, and comparisons are made with the preconditioned system. We have used three types of preconditioners:

- The diagonal preconditioner (matrix using diagonal entries of A),
- The additive Schwarz preconditioner based on domain decomposition (two-domain case),
- The multicative Schwarz preconditioner (symmetrized two-domain case).

In each case the convergence test used is `while relres > tol`, where `tol` $= 10^{-10}$ and `relres` $= \|r_m\|_2/\|r_0\|_2$ is the relative residual at each iteration step $m = 0, 1, 2, \ldots$ until convergence. For a clearer comparison we plot the graph of $\log(\|r_m\|_2/\|b\|_2)$ against the number of iterations m, where the vector b corresponds to the source function given by $f(x,y) = \pi^2\big(1 - 5\cos(2\pi y)\big)\sin(\pi x)$, $(x,y) \in \Omega$. The overlapping subproblems are also solved using conjugate gradient methods.

a) We solve the system using CG without precontioning, using $N = 100$. Figure 1 shows a plot of the logarithm of the relative residual $\log(\|r_m\|_2/\|b\|_2)$ as a function of the number of iterations m. In Table 1 we report the number of iterations required to

N	m
10	5
20	10
40	26
80	58
100	74
200	150

Table 1: Convergence of the CG. Number of iterations required to reduce the relative residual by 10 orders of magnitude.

Figure 2: Convergence behaviour of the additive PCG applied to the 2D Poisson problem with $N = 100$.

reduce the relative residual by 10 orders of magnitude for different values of $N = 10, 20, 40, 80, 100, 200$. Both the table and the figure show that the required number of iterations is $m \approx \mathrm{O}(N)$. This agrees with the predicted estimate. We also observe that the rate of convergence stays at approximately zero, but once the number of iterations reaches m, the relative residual drops spontaneously to the desired amount (i.e. $\leq 10^{-10}$).

b) Using the diagonal of A as a preconditioner we observe that there is no improvement in the rate of convergence. This is because the diagonal preconditioner in this case is given by $M = 4\mathrm{I}$ where I is the identity matrix, so that $\lambda(M^{-1}A) = \frac{1}{4}\lambda(A)$, and

$$\kappa_2(M^{-1}A) = \kappa_2\left(\tfrac{1}{4}A\right) = \frac{\frac{1}{4}\lambda_{\min}(A)}{\frac{1}{4}\lambda_{\max}(A)} = \kappa_2(A) = \kappa.$$

c) Now we use the additive Schwarz preconditioner. Figure 2 shows the convergence behaviour obtained with $N = 100$. In Table 2 we report the minimum number of iterations required to reduce the relative residual error by 10 orders of magnitude for $N = 10, 20, 40, 80, 100, 200$. The results show that with the additive Schwarz preconditioner, we have $m \approx \mathrm{O}(1)$. As N gets larger, the number of iterations required becomes *constant* ($m \approx 10$) independent of the mesh. This agrees with the theoretical result, namely that

$$\kappa(M^{-1}A) \approx \mathrm{O}\left(1 + \frac{H}{\delta}\right),$$

where $H \approx \mathrm{O}(h)$ is the dimension of the subdomain, and $\delta \approx \mathrm{O}(h)$ is the overlap amount.

d) Now we use the multiplicative Schwarz preconditioner discussed in Problem **2d)**. Figure 3 shows the convergence behaviour obtained with $N = 100$. In Table 3 we report the minimum number of iterations required to reduce the relative residual error by 10 orders of magnitude for different values of $N = 10, 20, 40, 80, 100, 200$.

Figure 3: Convergence behaviour of the multiplicative PCG applied to the 2D Poisson problem with $N = 100$.

N	m
10	6
20	8
40	9
80	10
100	10
200	10

Table 2: Convergence of the additive Schwarz PCG.

N	m
10	5
20	6
40	8
80	11
100	12
200	17

Table 3: Convergence of the multiplicative Schwarz PCG.

e) We observe from both Tables 2 and 3 that the multiplicative case performs slightly better than the additive case if the mesh resolution is not too dense, but if the resolution is higher (in this two-domain problem) the multiplicative case appears to require more iterations than the additive case. In either case, both overlapping Schwarz preconditioners significantly reduce the number of iterations required by the CG method to converge. One would only need to find more efficient ways for solving $M^{-1}r$.