Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4205 Numerical Linear Algebra Fall 2013

Solutions to exercise set 6

1 **a)** From the Householder process, we get the QR factorization

$$Q = \begin{bmatrix} -1/\sqrt{3} & -1/\sqrt{15} & 2/\sqrt{15} & -4/(5\sqrt{2}) \\ 1/\sqrt{3} & -2/\sqrt{15} & -1/\sqrt{15} & 5/(5\sqrt{2}) \\ 0 & 3/\sqrt{15} & -1/\sqrt{15} & 3/(5\sqrt{2}) \\ 1/\sqrt{3} & 1/\sqrt{15} & 3/\sqrt{15} & 0 \end{bmatrix}, \qquad R = \begin{bmatrix} \sqrt{3} & -1/\sqrt{3} & 0 \\ 0 & 5/\sqrt{15} & -3/\sqrt{15} \\ 0 & 0 & 6/\sqrt{15} \\ 0 & 0 & 0 \end{bmatrix}.$$

b) We find that

$$A^{\mathrm{T}}A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}.$$

By solving $\det(A^{\mathrm{T}}A - \lambda \mathrm{I}) = 0$, we find that the eigenvalues are

$$\lambda_1 = 4, \qquad \lambda_2 = 3, \qquad \lambda_3 = 1.$$

The corresponding eigenvectors x_i are obtained by finding solutions to $(A-\lambda_i \mathrm{I})x_i = 0$. The solutions are

$$x_1 = [1,-1,1]^{\mathrm{T}}, \qquad x_2 = [1,0,-1]^{\mathrm{T}}, \qquad x_3 = [1,2,1]^{\mathrm{T}}.$$

c) We want to find the SVD $A = U\Sigma V^{\mathrm{H}}$. The singular values of A are the square roots of the eigenvalues of $A^{\mathrm{T}}A$, sorted in descending order, so

$$\sigma_1 = 2, \qquad \sigma_2 = \sqrt{3}, \qquad \sigma_3 = 1.$$

The columns of V are obtained by normalizing the eigenvectors x_i:

$$v_1 = \frac{1}{\sqrt{3}}[1,-1,1]^{\mathrm{T}}, \qquad v_2 = \frac{1}{\sqrt{2}}[1,0,-1]^{\mathrm{T}}, \qquad v_3 = \frac{1}{\sqrt{6}}[1,2,1]^{\mathrm{T}}.$$

The first three columns of U are given by $u_i = \frac{1}{\sigma_i}Av_i$, so

$$u_1 = \frac{1}{\sqrt{3}}[0,1,-1,1]^{\mathrm{T}}, \qquad u_2 = \frac{1}{\sqrt{6}}[-2,1,1,0]^{\mathrm{T}}, \qquad u_3 = \frac{1}{\sqrt{6}}[0,-1,1,2]^{\mathrm{T}}.$$

The final column of U must be orthogonal to the first three and of unit length. We choose

$$u_4 = \frac{1}{\sqrt{3}}[1,1,1,0]^{\mathrm{T}}.$$

In conclusion, the SVD is

$$A = \begin{bmatrix} 0 & -2/\sqrt{6} & 0 & 1/\sqrt{3} \\ 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}.$$

d)

$$\begin{aligned}\|A\|_1 &= \text{max column sum} = 3,\\ \|A\|_2 &= \sigma_1 = 2,\\ \|A\|_\infty &= \text{max row sum} = 2,\\ \|A\|_\mathrm{F} &= \sqrt{\operatorname{tr}(A^\mathrm{H}A)} = 2\sqrt{2}.\end{aligned}$$

$\boxed{2}$ **a)** Let $Ax = \lambda x$. Then

$$(A - \mu \mathrm{I})x = Ax - \mu x = (\lambda - \mu)x,$$

so $\lambda - \mu$ is an eigenvalue of $A - \mu\mathrm{I}$.

b) This statement is false. To see this, let $A = \mathrm{I}$. Then $\lambda = 1$ is an eigenvalue of A, but -1 is not.

c) Let $Ax = \lambda x$, where A is real. Then simply take the complex conjugate of this equation, to get

$$A\bar{x} = \bar{\lambda}\bar{x},$$

so $\bar{x}$ is an eigenvalue of A.

d) Let $Ax = \lambda x$, where A is nonsingular. Then $\lambda \neq 0$, so we can divide by λ and multiply by A^{-1} from the left to obtain

$$A^{-1}x = \lambda^{-1}x.$$

Thus λ^{-1} is an eigenvalue of A^{-1}.

e) This statement is false. For example, all triangular matrices with zeros on the diagonal only have 0-eigenvalues. One such matrix is

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

f) Since $A^\mathrm{H} = A$, we can unitarily diagonalize $A = Q\Lambda Q^\mathrm{H}$. Then

$$\sigma = \sqrt{\lambda(A^\mathrm{H}A)} = \sqrt{\lambda(Q\Lambda^2 Q^\mathrm{H})} = \sqrt{\lambda(A)^2} = |\lambda|.$$

g) Let $A = X\Lambda X^{-1}$. If all the eigenvalues are equal to λ, then $\Lambda = \lambda\mathrm{I}$, so

$$A = X\lambda\mathrm{I}X^{-1} = \lambda\mathrm{I},$$

which is diagonal.