TMA4205 Numerical
Linear Algebra
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Norwegian University of Science and
Technology
Department of Mathematical
Sciences

Exercise set 2

1 We shall study some properties of positive definite matrices.

    a) Prove that the following two statements are equivalent for a normal matrix $A \in \mathbb{R}^{n \times n}$:

- The matrix A is positive definite.
- All the eigenvalues of A are positive.

    b) Prove that the condition number based on the Euclidean norm of a normal matrix A, can be written as

$$\kappa(A) = \frac{\max_i |\lambda_i|}{\min_i |\lambda_i|},$$

    where λ_i are the eigenvalues of A.

    c) Prove that a matrix A is positive definite if and only if A^{-1} is positive definite.

    d) Let $A \in \mathbb{R}^{n \times n}$ and let $x \in \mathbb{R}^n$. The expression

$$R(x) = \frac{x^{\mathrm{T}} A x}{x^{\mathrm{T}} x}$$

    is called the Rayleigh quotient of A. If A is normal and positive definite, show that

$$\lambda_1 \leq R(x) \leq \lambda_n \quad \text{for all } x \in \mathbb{R}^n \setminus \{0\},$$

    where λ_1 and λ_n are the smallest and largest eigenvalue of A respectively.

2 Consider the matrix

$$A = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}.$$

    a) Is A normal?

    b) Find the eigenvalues and eigenvectors of A.

    c) Are the eigenvectors linearly independent? Are they orthogonal?

    d) Can A be diagonalized?

    e) Show that there exists a constant $\alpha > 0$ so that $u^{\mathrm{T}} A u \geq \alpha \|u\|_2^2$ for all $u \in \mathbb{R}^2$. What is the largest possible value for α?

    f) Is A positive definite?

    g) Find a Schur factorization for A.

3 We consider again the Poisson problem

$$-\Delta u = f, \quad \text{in } \Omega = (0,1) \times (0,1),$$
$$u = 0, \quad \text{on } \partial\Omega.$$

We discretize the system on a uniform grid with step-size $h = 1/n$ in each direction.

a) Solve the resulting system of linear equations in MATLAB with three different methods:

   i) The diagonalization method discussed in Einar Rønquist's note.

   ii) LU-factorization with sparse matrices.

  iii) Full LU-factorization without sparse matrices.

b) Do the timings scale as expected?

c) Calculate the Euclidean condition number $\kappa(A)$ of the discrete Laplacian A for various different values of n. How does $\kappa(A)$ scale with n?

d) How does $\kappa(A)$ scale for the one-dimensional Laplacian? Compare these results with the results from question **c)**.