Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4205 Numerical Linear Algebra
Fall 2014

Exercise set 4

1 **Saad, Exercise 5.3** In Section 5.3.3, it was shown that using a one-dimensional projection method with $\mathcal{K} = \text{span}\{A^T r\}$ and $\mathcal{L} = \text{span}\{AA^T r\}$ is equivalent to using the steepest descent method on the normal equations $A^T Ax = A^T b$.

Show that an *orthogonal* projection method for $A^T Ax = A^T b$ with search space $\mathcal{K}$ is equivalent to applying a projection method onto $\mathcal{K}$ orthogonally to $\mathcal{L} = A\mathcal{K}$ for the problem $Ax = b$.

2 Algorithm 6.1 in Saad is implemented in the MATLAB-function `arnoldi_gs.m`. This algorithm constructs an orthogonal basis for the Krylov subspace $\mathcal{K}_m(A, v)$ based on a classical Gram–Schmidt procedure. Test this function on the matrix A generated by `poisson2.m` (use sparse matrices) for different values of m and $N = n^2$. For instance, choose $N = 100$, $v = \mathrm{e}_1$, and $m = 10, 20, 30, 40$.

a) Test to what extent the relation $V_m^T AV_m = H_m$ from Proposition 6.5 in Saad is fulfilled. Also check if the vectors $v_1, \dots, v_m$ really are orthonormal, i.e. check whether $V_m^T V_m = \mathrm{I}_m$.

b) Modify the function `arnoldi_gs.m` such that it uses modified Gram–Schmidt. Repeat the experiments from the previous question.

3 Implement a Matlab program for finding an orthogonal basis for the Krylov subspace $\mathcal{K}_m(A, v)$ in a "naïve" fashion. That is, given A, v, first generate an $n \times m$ matrix K with $K_{*1} = v/\|v\|_2$ and $K_{*i} = AK_{*i-1}/\|AK_{*i-1}\|_2$ of normalized vectors, which span $\mathcal{K}_m(A, v)$. Then find a "tall" QR-factorization of $K = V_m R_m$; this produces an orthogonal basis $V_m = [v_1, \dots, v_m]$ for $\mathcal{K}_m(A, v)$.

Use the matrix A generated by `poisson2.m` (use sparse matrices). Take $N = n^2 = 100$, $m = 50$, $v = \mathrm{e}_1$.

Compute the rank of the matrices K and R_m; do they predict the dimension of $\mathcal{K}_m(A, v)$ correctly? Plot the absolute values of the diagonal elements in R on a logarithmic scale.

4 Implement Householder reflection-based Arnoldi iteration `arnoldi_h.m` in Matlab with the same interface as `arnoldi_gs.m` (see Algorithm 6.3 in Saad). Repeat the numerical experiment of the exercise 2a) and compare the results.