Norwegian University of Science and Technology
Department of Mathematical Sciences

Exercise set 5

1 Assume that a real matrix A is anti-symmetric, that is, $A^{\mathrm{T}} = -A$. Explain the structure of the Hessenberg matrix H_m resulting from Arnoldi process in this case. Explain how this structure can be utilized for performing Arnoldi process efficiently in this case.

2 **a)** If A is symmetric and positive definite (SPD), show that A^{-1} can be used to define a norm on $\mathbb{R}^n$,

$$\|v\|_{A^{-1}} = (v^{\mathrm{T}} A^{-1} v)^{1/2}$$

b) We know that the conjugate gradient (CG) method will minimize the error in A-norm over all elements in the Krylov subspace $\mathcal{K}_m(A, r_0)$. Show that the algorithm also, in each iteration, will minimize the associated residual in A^{-1}-norm.

c) Each update of the solution in CG can be expressed as $x_{j+1} = x_j + \alpha_j p_j$, where $\alpha_j = (r_j, r_j)/(Ap_j, p_j)$ (see Algorithm 6.18 in Saad). Show that α_j is optimal in the sense that it minimizes the functional $f(w) = \frac{1}{2} w^{\mathrm{T}} A w - w^{\mathrm{T}} b$, $f\colon \mathbb{R}^n \to \mathbb{R}$, along the search direction p_j.

3 **Saad, Exercise 6.4**.