Norwegian University of Science and Technology
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TMA4205 Numerical Linear Algebra
Fall 2014

Exercise set 6

$\boxed{1}$ Consider a two-diagonal matrix $A \in \mathbb{R}^{n\times n}$

$$A = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 & 1 \end{pmatrix},$$

and let e_i denote the ith canonical basis vector in $\mathbb{R}^n$. Let $b = e_1$, $x_0 = 0$.

a) Verify that A is non-singular and find $x^* \in \mathbb{R}^n$ solving the system $Ax = b$.

b) Compute the residual r_0 and prove that $K_m(A, r_0) = \operatorname{span}\langle e_1, \dots, e_m\rangle$, $1 \le m \le n$.

c) Show that any Krylov subspace method for this problem starting from x_0 must satisfy the lower error bound $\|x_m - x^*\|_2^2 \ge n - m$, $0 \le m \le n$. Show that a similar error bound (up to a constant C_n depending on n) is satisfied by the residuals: $\|r_m\|_2^2 \ge C_n(n-m)$.

d) Let $n = 5$. Using the optimality property of GMRES

$$\begin{aligned}\|r_m\|_2 &= \min_{x\in x_0+\mathcal{K}_m(A,r_0)} \|b - Ax\|_2 = \min_{p_{m-1}\in\mathbb{P}_{m-1}} \|[I - Ap_{m-1}(A)]r_0\|_2 \\ &= \min_{\tilde{p}_m\in\mathbb{P}_m : \tilde{p}_m(0)=1} \|\tilde{p}_m(A)r_0\|_2,\end{aligned}$$

numerically (using Matlab) find the polynomials $\tilde{p}_i(t)$, $i = 1,\dots,5$ and plot them on the same graph.
FInally, numerically compute $\|\tilde{p}_i(A)\|_2$ and $\|r_i\|_2 = \|\tilde{p}_i(A)r_0\|_2$.

e) Repeat the previous point, but for the matrix $(A + A^{\mathrm{T}})/2$. Compute the spectrum of A (respectively, $(A + A^{\mathrm{T}})/2$) and compare the behaviour of the minimal polynomials. Explain why eigenvalue-based error bounds for Krylov subspace methods do not apply to A.

$\boxed{2}$ Assume that $A \in \mathbb{R}^{n\times n}$ is a diagonalizable non-singular matrix with real positive eigenvalues distributed on an interval $0 < \lambda_{\min} \le \lambda_i \le \lambda_{\max}$, $i = 1,\dots,n-1$ and one “very different” (for example negative, or very small/very large) eigenvalue $0 \ne \lambda_n = \bar{\lambda} \not\in [\lambda_{\min}, \lambda_{\max}]$.

Construct an upper estimate for the quantity $\|r_m\|_2/\|r_0\|_2$ after $m > 1$ iterations of GMRES using the following idea: take $\tilde{p}_m(\lambda) = a_m C_{m-1}(t(\lambda))(\lambda - \bar{\lambda})$, where a_m is the renormalization constant such that $\tilde{p}_m(0) = 1$, C_{m-1} is the Chebyshev polynomial of degree $m-1$, and $t : [\lambda_{\min}, \lambda_{\max}] \to [-1, 1]$ is an affine map.

Conclude that at least in the case of normal A and $|\bar{\lambda}| >> \max\{\lambda_{\min}, \lambda_{\max}\}$ we can expect that the Krylov method requires at most one additional iteration to obtain similar accuracy as the method applied to a matrix with eigenvalues in $[\lambda_{\min}, \lambda_{\max}]$.

3 Show that both CG and GMRES are scaling independent. That is, when applied to a system $(\delta A)x = \delta b$ for some $\delta \in \mathbb{R} \setminus \{0\}$ ($\delta > 0$ in the case of CG) they produce exactly the same sequence of iterates in exact arithmetics. (Of course since the residual vectors will be scaled by δ this may affect stopping criteria of the methods.)