

## Uniqueness of SVD:

Let  $A$  be a square matrix with  $n$  distinct singular values  $\sigma_1 > \sigma_2 > \dots > \sigma_n \geq 0$ . Then  $U, V, \Sigma$  are uniquely determined [ $U, V$ - up to complex signs, i.e., complex multipliers of magnitude 1].

Proof Let  $A = U\Sigma V^*$  be some SVD of  $A$ .

$$\text{Then } AA^* = U\Sigma V^* V U^* = U\Sigma^2 U^*, \quad \Sigma^2 = \Sigma^2.$$

Thus  $\lambda_i = \sigma_i^2$  are distinct as well  $\Rightarrow$  algebraic multiplicity of  $\lambda_i$  is 1  $\Rightarrow$  geometric mult. of  $\lambda_i$  is 1.

Thus  $AA^* u_i = \lambda_i u_i$  has a "unique" solution (ie determined up to complex scalar multipliers).

Since eigenvalues of  $AA^*$  are uniquely determined (as roots of  $\det(A - \lambda I) = 0$ )  $\Rightarrow \sigma_i = \sqrt{\lambda_i}$  is uniquely determined as well.

Uniqueness of  $V$  is obtained by considering  $A^*A$  instead of  $AA^*$ .