Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4205 Numerical Linear Algebra Fall 2014

Solutions to exercise set 1

1 **a)** We have that

$$\|x\|_1^2 = \Big(\sum_{i=1}^n |x_i|\Big)^2 = \sum_{i=1}^n \sum_{j=1}^n |x_i||x_j| = \sum_{i=1}^n |x_i|^2 + \sum_{i=1}^n \sum_{j\neq i} |x_i||x_j| \geq \|x\|_2^2,$$

since $|x_i||x_j|$ is non-negative. We also have

$$\begin{aligned}
\|x\|_1^2 &= \Big(\sum_{i=1}^n |x_i|\Big)^2 \\
&= \sum_{i=1}^n \sum_{j=1}^n |x_i||x_j| \\
&\leq \tfrac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (|x_i|^2 + |x_j|^2) \qquad \big(ab \leq \tfrac{1}{2}(a^2+b^2) \text{ for any } a, b \in \mathbb{R}\big) \\
&= n \sum_{i=1}^n |x_i|^2 \\
&= n\|x\|_2^2.
\end{aligned}$$

Hence, $\|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\,\|x\|_2$. Alternatively, for the second step we could have used the Cauchy–Schwarz inequality

$$|y^{\mathrm{H}} x| \leq \|x\|_2 \|y\|_2$$

and chosen

$$y_i = \begin{cases} 1 & x_i = 0, \\ x_i/|x_i| & x_i \neq 0. \end{cases}$$

Since $|y_i| = 1$, this would have given $\|x\|_1 = |y^{\mathrm{H}} x| \leq \|x\|_2 \|y\|_2 = \sqrt{n}\,\|x\|_2$, giving the same result as above.

b) For the first inequality,

$$\|x\|_2^2 = \sum_{i=1}^n |x_i|^2 \geq \max_{1\leq i\leq n} |x_i|^2 = \|x\|_\infty^2.$$

For the second inequality we have

$$\|x\|_2^2 = \sum_{i=1}^n |x_i|^2 \leq \sum_{i=1}^n \max_{1\leq j\leq n} |x_j|^2 = \sum_{i=1}^n \|x\|_\infty^2 = n\|x\|_\infty^2.$$

Hence, $\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\,\|x\|_\infty$.

c)

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| \geq \max_{1\leq i\leq n} |x_i| = \|x\|_\infty$$

and

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| \leq \sum_{i=1}^{n} \max_{1\leq j\leq n} |x_j| = n\|x\|_\infty$$

give us that $\|x\|_\infty \leq \|x\|_1 \leq n\|x\|_\infty$.

$\boxed{2}$ ($\Rightarrow$) Assume that A has full rank. Then no column of A can be written as a linear combination of the remaining columns. That is, if a_j denotes the jth column of A, there exists no vector $c \in \mathbb{C}^n$ such that

$$a_j = \sum_{\substack{i=1\\ i\neq j}}^{n} c_i a_i, \quad j = 1,\ldots,n.$$

Thus, if A has full rank, $Ac = 0$ implies $c = 0$. This means that A has a trivial nullspace (i.e. $\operatorname{Ker} A = \{0\}$).

Let x_1 and x_2 be two distinct vectors. Then $x_1 - x_2 \neq 0$, and

$$Ax_1 - Ax_2 = A(x_1 - x_2) \neq 0, \quad \text{since } \operatorname{Ker} A = \{0\}.$$

Thus, Ax_1 and Ax_2 are distinct vectors.

($\Leftarrow$) For the sake of contradiction, assume that A does not have a full rank. Then for some $j = 1,\ldots,n$ and some $c \in \mathbb{C}^{n-1}$ it holds that

$$a_j = \sum_{\substack{i=1\\ i\neq j}}^{n} c_i a_i, \quad j = 1,\ldots,n.$$

For convenience we put $c_j = -1$. Then we can succintly write the previous equation as $Ac = 0$. But $A0 = 0$ as well. Thus the matrix A maps at least two distinct vectors 0, $c \neq 0$ into the same vector 0.

$\boxed{3}$

$$[b_1 \mid \cdots \mid b_n] = [Ar_1 \mid \cdots \mid Ar_n], \qquad R = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ & 1 & \cdots & 1 \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix}$$

We have that

$$b_{i,j} = \sum_{k=1}^{n} a_{i,k} r_{k,j},$$

but since

$$r_{i,j} = \begin{cases} 1 & i \leq j, \\ 0 & i > j, \end{cases}$$

we get

$$b_{i,j} = \sum_{k=1}^{j} a_{i,k} \quad \Longrightarrow \quad b_j = \sum_{k=1}^{j} a_k.$$

4 We define the matrix norm induced by the vector norm $\|\cdot\|$ by

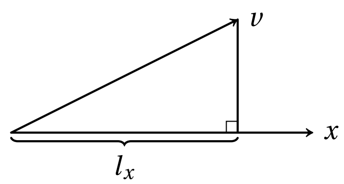
$$\|A\| := \max_{x\in\mathbb{C}^n\setminus\{0\}} \frac{\|Ax\|}{\|x\|}.$$

Suppose A has p distinct eigenvalues $\lambda_1,\dots,\lambda_p$ corresponding to eigenvectors $v_1,\dots,v_p$ respectively. Then

$$\|A\| \geq \max_i \frac{\|Av_i\|}{\|v_i\|} = \max_i \frac{|\lambda_i|\|v_i\|}{\|v_i\|} = \max_i |\lambda_i| = \rho(A).$$

5 **a)** We have

$$\begin{aligned}
\|E\|_2 &= \max_{x\in\mathbb{C}^n\setminus\{0\}} \frac{\|uv^{\mathrm{H}}x\|_2}{\|x\|_2} \\
&= \max_{x\in\mathbb{C}^n\setminus\{0\}} \frac{|v^{\mathrm{H}}x|\|u\|_2}{\|x\|_2} \\
&= \|u\|_2 \max_{x\in\mathbb{C}^n\setminus\{0\}} \frac{|v^{\mathrm{H}}x|}{\|x\|_2} \\
&= \|u\|_2 \frac{|v^{\mathrm{H}}v|}{\|v\|_2} \\
&= \|u\|_2\|v\|_2.
\end{aligned}$$



(Maximum is attained when $x = \alpha v$, for $\alpha \neq 0$ scalar.)

Note that $l_x = |v^{\mathrm{H}}x|/\|x\|_2$ represents the length of the orthogonal projection of v on x, so l_x is maximal for vectors x having the *same direction* as v.

b) For the Frobenius norm we get

$$\|E\|_{\mathrm{F}} = \sqrt{\sum_{i=1}^n\sum_{j=1}^n |u_i\bar{v}_j|^2} = \sqrt{\sum_{i=1}^n |u_i|^2 \sum_{j=1}^n |v_j|^2} = \sqrt{\sum_{i=1}^n |u_i|^2}\sqrt{\sum_{j=1}^n |v_j|^2} = \|u\|_{\mathrm{F}}\|v\|_{\mathrm{F}}.$$

6 For a unitary matrix Q, $Q^{\mathrm{H}}Q = \mathrm{I}$, where I is the identity matrix. We recall that the 2-norm of a vector is given by $\|v\|_2 = \sqrt{v^{\mathrm{H}}v}$. This gives

$$\|Qx\|_2 = \sqrt{(Qx)^{\mathrm{H}}Qx} = \sqrt{x^{\mathrm{H}}Q^{\mathrm{H}}Qx} = \sqrt{x^{\mathrm{H}}x} = \|x\|_2.$$

Also, let φ_i be an eigenvector of Q with corresponding eigenvalue λ_i. Then

$$\|Q\varphi_i\|_2 = \|\varphi_i\|_2. \tag{1}$$

But

$$\|Q\varphi_i\|_2 = \|\lambda_i\varphi_i\|_2 = |\lambda_i|\|\varphi_i\|_2. \tag{2}$$

Comparing (1) and (2) we get that

$$|\lambda_i| = 1.$$

Thus, $\rho(Q) = 1$.

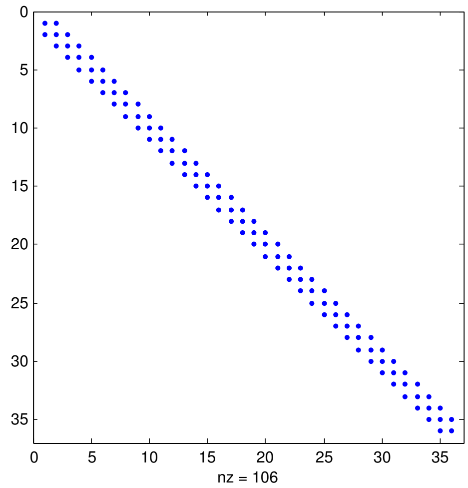


Figure 1: Nonzero entries in the 1D Poisson matrix for $n = 36$

7 The purpose of this exercise is to show how the storage format and structure of a matrix can influence the performance of the LU factorization of the matrix.

a) We here consider the one-dimensional problem

$$-\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = f(x), \quad x \in [0,1],$$
$$u = 0, \qquad x \in \{0,1\}.$$

We discretize the problem using finite differences, applied on a partition of the domain $[0,1]$ into $n+1$ subintervals. We get the difference equations

$$-\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} = f_i, \quad 1 \le i \le n,$$
$$u_0 = u_{n+1} = 0,$$

where $h = 1/(n+1)$ and u_i is the numerical approximation of $u(ih)$. Since the difference equations relate three neighbouring points, the resulting matrix must be tridiagonal (an $n \times n$ banded matrix with bandwidth 1). See Figure 1.

With Gaussian elimination we eliminate the entries below the diagonal. For a full matrix this would require $\mathrm{O}(n^3)$ flops, but for a one-dimensional Poisson problem we only have to eliminate the entries in the first sub-diagonal, since the rest are zero. This would require only $\mathrm{O}(n)$ flops. Generally for a banded matrix of bandwidth k, we must eliminate all nonzero entries below the diagonal. This would require $\mathrm{O}(nk^2)$ flops.

i) For $n \times n$ matrices, Gaussian elimination in MATLAB is done by LU factorization. When we save the matrix as a full matrix, MATLAB will eliminate all entries under the diagonal, including entries that are already zeros. The amount of computing time will be $\mathrm{O}(n^3)$. That means, doubling the problem dimension would increase the computing time by a factor 8. This can be observed in Table 1. Since the predicted computing time is $\mathrm{O}(n^3)$ we observe that Time$/n^3$ is approximately constant ($\approx 1 \cdot 10^{-11}$).[1] This confirms that the computing time is $\mathrm{O}(n^3)$.

ii) We repeat the numerical experiment in **i)** but using the sparse format in MATLAB for storing the matrix. Here we observe $\mathrm{O}(n)$ computing time (see Table 2).

[1] Alternatively, a `loglog` plot of time versus n would give a straight line whose slope is approximately equal to 3.

n	Time (s)	Time/n^3
900	0.0104	$1.43 \cdot 10^{-11}$
1600	0.0443	$1.08 \cdot 10^{-11}$
2500	0.1551	$9.929 \cdot 10^{-12}$
3600	0.4090	$8.766 \cdot 10^{-12}$

Table 1: One-dimensional Poisson problem with full matrix

n	Time (s)	Time/n
900	$1.81 \cdot 10^{-4}$	$2.01 \cdot 10^{-7}$
1600	$3.11 \cdot 10^{-4}$	$1.94 \cdot 10^{-7}$
2500	$4.83 \cdot 10^{-4}$	$1.93 \cdot 10^{-7}$
3600	$6.93 \cdot 10^{-4}$	$1.93 \cdot 10^{-7}$

Table 2: One-dimensional Poisson problem with sparse matrix

b) We consider the two-dimensional problem

$$
\begin{aligned}
-\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) &= f(x,y), \qquad (x,y) \in [0,1] \times [0,1], \\
u &= 0, \qquad x = 0, x = 1, y = 0, y = 1.
\end{aligned}
$$

We partition $[0,1]$ into $n+1$ intervals in both the x- and y-directions, and obtain the following difference equations upon discretizing the problem:

$$
\begin{aligned}
-\frac{u_{i,j-1} + u_{i-1,j} - 4u_{i,j} + u_{i+1,j} + u_{i,j+1}}{h^2} &= f, \qquad 1 \le i, j \le n, \\
u_{0,j} = u_{n+1,j} = u_{i,0} = u_{i,n+1} &= 0, \qquad 0 \le i, j \le n+1,
\end{aligned}
$$

where $h = 1/(n+1)$ and $u_{i,j}$ is the numerical approximation of $u(x_i, y_j)$ with $x_i = ih$ and $y_j = jh$. Each difference equation relates the three neighbouring points in the x-direction and the three neighbouring points in the y-direction. We number the unknown variables first in the x-direction from bottom to top. The resulting matrix has almost the same structure as in **a)**, but in addition we have two bands a distance n away from the diagonal. See Figure 2.

The number of unknowns has now become $N = n^2$, and the bandwidth $k = 2n+1 = 2\sqrt{N}+1$. Storing the matrix as a full matrix will require that the computing time increases as $O(N^3)$, but if we store it as a sparse matrix we predict a computing time of order $O(Nk^2) = O(N^2)$.

- **i)** Se Table 3 for the computing time with full matrix. We observe that the result is like in **a)**, with computing time $O(N^3)$.
- **ii)** We repeat the experiment using MATLAB's sparse storage format (see Table 4). Indeed, we observe $O(N^2)$ computing time here. The matrix structure in the 2D case is different from the 1D case (for the same number of unknowns), but by exploiting sparsity, we get a significant speedup here too.

This exercise illustrates the problems encountered with using Gaussian elimination in the numerical solution of differential equations in several space-dimensions (e.g. 2D or 3D). The bandwidth increases with space-dimension even if the dimension of the linear

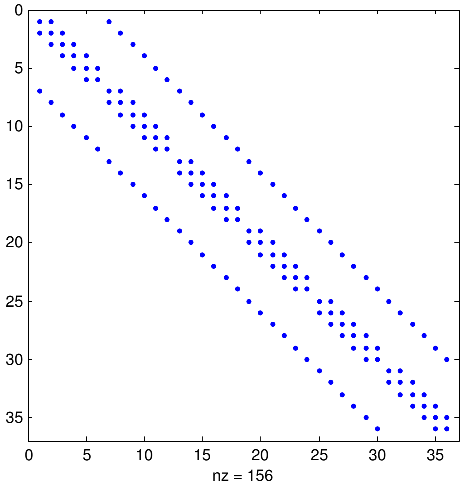


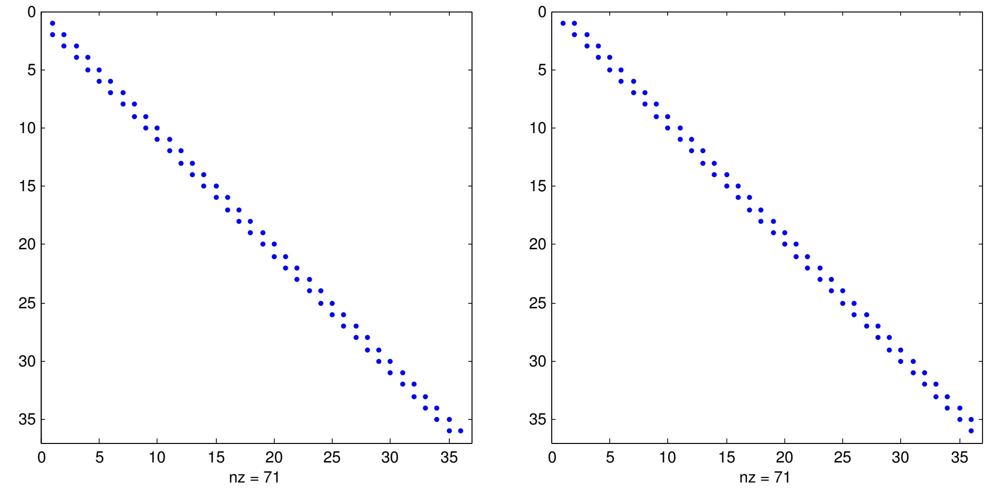
Figure 2: Nonzero entries in the 2D Poisson matrix for $n = 6$.

N	Time (s)	Time$/N^3$
900	0.0097	$1.3 \cdot 10^{-11}$
1600	0.0428	$1.04 \cdot 10^{-11}$
2500	0.1526	$9.765 \cdot 10^{-12}$
3600	0.4051	$8.682 \cdot 10^{-12}$

Table 3: Two-dimensional Poisson problem with full matrix

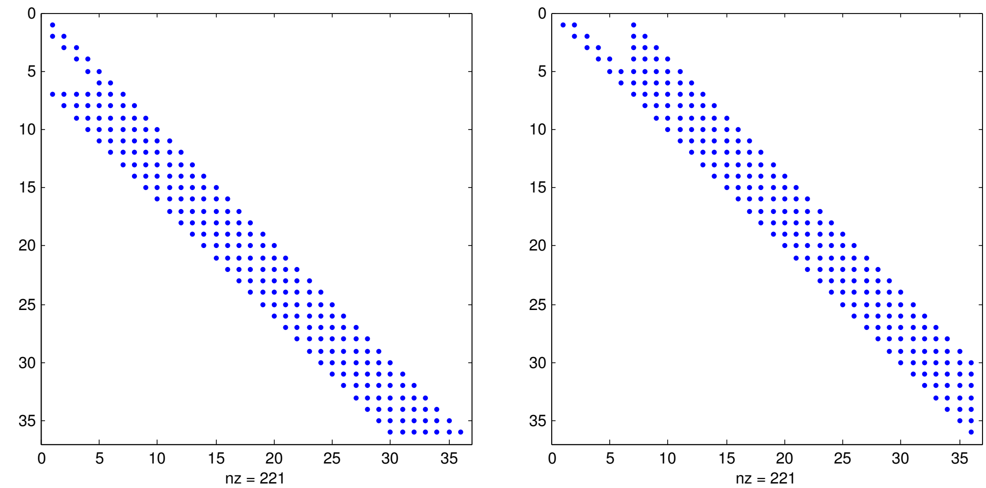
N	Time (s)	Time$/N^2$
900	$2.242 \cdot 10^{-3}$	$2.768 \cdot 10^{-9}$
1600	$6.824 \cdot 10^{-3}$	$2.666 \cdot 10^{-9}$
2500	$1.552 \cdot 10^{-2}$	$2.484 \cdot 10^{-9}$
3600	$2.979 \cdot 10^{-2}$	$2.298 \cdot 10^{-9}$

Table 4: Two-dimensional Poisson problem with sparse matrix



(a) L (b) U

Figure 3: Nonzero entries for L and U after LU factorization of the 1D Poisson matrix.



(a) L (b) U

Figure 4: Nonzero entries for L and U after LU factorization of the 2D Poisson matrix.

system is kept constant. This makes Gaussian elimination less attractive for the solution of large linear systems even with sparse matrix format. After LU-factorization the storage requirement also increases, especially in the case of higher space-dimension. There are $O(n)$ nonzero entries before LU-factorization. After LU-factorization nearly all entries between the diagonal and the outermost sub- or super-diagonal are nonzero. The storage increases to about $O(nk)$. See Figures 3 and 4.