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TMA4205 Numerical Linear Algebra
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Solutions to exercise set ?

$\boxed{1}$ Exercise 5.7 in Demmel.

Given: $q \perp d$, $\|q\|_2 = 1$, $B := (q+d)q^{\mathrm{T}} - I$. Show: $\|B\|_2 = \|q+d\|_2$.

If $d = 0$ then $B = qq^{\mathrm{T}} - I$ is a projector, and $\|B\|_2 = 1 = \|q\|_2 = \|q+d\|_2$. From now on we will assume that $\|d\|_2 \neq 0$.

We will find the largest eigenvalue of $B^{\mathrm{T}}B$, which is equal to $\|B\|_2^2$.

$$\begin{aligned} B^{\mathrm{T}}B &= \|q+d\|_2^2 qq^{\mathrm{T}} - (q+d)q^{\mathrm{T}} - q(q+d)^{\mathrm{T}} + I \\ &= (\|q+d\|_2^2 - 2)qq^{\mathrm{T}} - dq^{\mathrm{T}} - qd^{\mathrm{T}} + I. \end{aligned}$$

Take $x \in \operatorname{span}(q,d)^{\perp} =: L$; then $q^{\mathrm{T}}x = d^{\mathrm{T}}x = 0$ and $B^{\mathrm{T}}Bx = Ix = x$. Thus L is a $n-2$ dimensional eigenspace of $B^{\mathrm{T}}B$, corresponding to the eigenvalue 1. Let us look at $L^{\perp}$ now, which is a 2-dimensional space spanned by the orthonormal basis q and $d/\|d\|_2$.

Let us compute the matrix $B^{\mathrm{T}}B$ restricted to $L^{\perp}$ in the basis q and $d/\|d\|_2$. We take $x = \alpha q + \beta d/\|d\|_2$.

$$\begin{aligned} B^{\mathrm{T}}Bx &= \alpha(\|q+d\|_2^2 - 2)q - \alpha d - \beta\|d\|_2 q + \alpha q + \beta d/\|d\|_2 \\ &= [\|q+d\|_2^2 - 1) - \beta\|d\|_2]q + [-\alpha\|d\|_2 + \beta]d/\|d\|_2. \end{aligned}$$

As a result, $\forall x = \alpha q + \beta d/\|d\|_2$:

$$B^{\mathrm{T}}Bx = \left\{[q, d/\|d\|_2] \underbrace{\begin{pmatrix} \|q+d\|_2^2 - 1 & -\|d\|_2 \\ -\|d\|_2 & 1 \end{pmatrix}}_{=:\hat{M}} [q, d/\|d\|_2]^{\mathrm{T}}\right\}[q, d/\|d\|_2]\begin{pmatrix}\alpha \\ \beta\end{pmatrix}$$

Taking into account that $\|q+d\|_2^2 = \|q\|_2^2 + \|d\|_2^2 = 1 + \|d\|_2^2$ owing to the orthogonality of q and d, we compute $\det\hat{M} = 0 = \lambda_1\lambda_2$ and $\operatorname{tr}\hat{M} = \|q+d\|_2^2 = \lambda_1 + \lambda_2$, thus resulting in $\lambda_1 = \|q+d\|_2^2 = 1 + \|d\|_1^2 \geq 1$ and $\lambda_2 = 0$. As a result, $\|B\|_2 = \lambda_1^{1/2} = \|q+d\|_2$. □

$\boxed{2}$ Exercise 5.4 in Demmel. (Cauchy interlacing theorem).

Given $A \in \mathbb{R}^{n\times n}$-symmetric matrix, where

$$A = \begin{pmatrix} H & b \\ b^{\mathrm{T}} & u \end{pmatrix},$$

where $H \in \mathbb{R}^{(n-1)\times(n-1)}$ is a symmetric submatrix of A. Then, the eigenvalues $\alpha_n \leq \cdots \leq \alpha_1$ of A and eigenvalues $\theta_{n-1} \leq \cdots \leq \theta_1$ of H are *interlaced*, that is

$$\alpha_n \leq \theta_{n-1} \leq \cdots \leq \theta_i \leq \alpha_i \leq \theta_{i-1} \leq \alpha_{i-1} \leq \cdots \leq \theta_1 \leq \alpha_1.$$

Proof: For any $x \in \mathbb{R}^n$ we will write $x = (x_1, x_2)$ where $x_1 \in \mathbb{R}^{n-1}$ are the first $n-1$ components of x and $x_2 \in \mathbb{R}$ is the last one.

First, we will show that $\theta_i \le \alpha_i$, $i = 1, \ldots, n-1$. Indeed, for any subspace S of $\mathbb{R}^n$ with $\dim S \ge 2$ we have

$$\max_{x \in S \setminus \{0\}} \frac{x^{\mathrm{T}} A x}{x^{\mathrm{T}} x} \ge \max_{x \in S \setminus \{0\}, x_2 = 0} \frac{x^{\mathrm{T}} A x}{x^{\mathrm{T}} x} = \max_{x \in S \setminus \{0\}, x_2 = 0} \frac{x_1^{\mathrm{T}} H x_1}{x_1^{\mathrm{T}} x_1},$$

because maximum is taken over a smaller set on the right hand side of the inequality. Therefore, owing to Courant–Fischer minmax theorem:

$$\begin{aligned}
\alpha_i &= \min_{S \subseteq \mathbb{R}^n : \dim S = n-i+1} \max_{x \in S \setminus \{0\}} \frac{x^{\mathrm{T}} A x}{x^{\mathrm{T}} x} = \min_{S \subseteq \mathbb{R}^n : \dim S \ge n-i+1} \max_{x \in S \setminus \{0\}} \frac{x^{\mathrm{T}} A x}{x^{\mathrm{T}} x} \\
&\ge \min_{S \subseteq \mathbb{R}^n : \dim S \ge n-i+1} \max_{x \in S \setminus \{0\}, x_2 = 0} \frac{x_1^{\mathrm{T}} H x_1}{x_1^{\mathrm{T}} x_1} = \min_{S_1 \subseteq \mathbb{R}^{n-1} : \dim S_1 \ge n-i} \max_{x_1 \in S_1 \setminus \{0\}} \frac{x_1^{\mathrm{T}} H x_1}{x_1^{\mathrm{T}} x_1} \\
&= \min_{S_1 \subseteq \mathbb{R}^{n-1} : \dim S_1 = n-i} \max_{x_1 \in S_1 \setminus \{0\}} \frac{x_1^{\mathrm{T}} H x_1}{x_1^{\mathrm{T}} x_1} = \theta_i,
\end{aligned}$$

$i = 1, \ldots, n-1$. Note that as $x = (x_1, x_2)$ lives in an arbitrary subset of $\mathbb{R}^n$ of dimension at least $n-i+1$, its first $n-1$ components live in an arbitrary subset S_1 of $\mathbb{R}^{n-1}$ of dimension at least $n-i$.

We will now show that $\alpha_i \le \theta_{i-1}$, $i = 2, \ldots, n$.

$$\begin{aligned}
\alpha_i &= \min_{S \subseteq \mathbb{R}^n : \dim S = n-i+1} \max_{x \in S \setminus \{0\}} \frac{x^{\mathrm{T}} A x}{x^{\mathrm{T}} x} \le \min_{S \subseteq \mathbb{R}^n : \dim S = n-i+1, S \perp e_n} \max_{x \in S \setminus \{0\}} \frac{x^{\mathrm{T}} A x}{x^{\mathrm{T}} x} \\
&= \min_{S \subseteq \mathbb{R}^n : \dim S = n-i+1, S \perp e_n} \max_{x \in S \setminus \{0\}} \frac{x_1^{\mathrm{T}} H x_1}{x_1^{\mathrm{T}} x_1} \\
&= \min_{S_1 \subseteq \mathbb{R}^{n-1} : \dim S_1 = n-i+1} \max_{x_1 \in S_1 \setminus \{0\}} \frac{x_1^{\mathrm{T}} H x_1}{x_1^{\mathrm{T}} x_1} = \theta_{i-1}.
\end{aligned}$$

The inequality arises because the minimum is taken over a restricted set of subspaces (if $i = 1$ then the only choice for S is $\mathbb{R}^n$ and therefore it is impossible to select $S \perp e_n$). Now as $x = (x_1, 0)$ belongs to an $n-i+1$-dimensional subset $S \perp e_n$ of $\mathbb{R}^n$, then x_1 belongs to $n-i+1$-dimensional subset of $\mathbb{R}^{n-1}$; this is different in the previous string of inequalities!